Distributed Control of Autonomous Mobile Robots

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Chapter 1

Introduction

The subject of the course is the coordinated control of autonomous mobile robots. Autonomous means that the robot has only sensory inputs—no outside direct commands. We shall restrict to distributed control: Each robot has the same local strategy—no leaders. Finally, we restrict to decentralized control: At least one robot cannot sense all others.

1.1 Motivation

The problem of coordinated control of a network of mobile autonomous robots is of interest in control and robotics because of the broad range of potential applications: planetary exploration, operations in hazardous environments, etc. Distributed robot networks can potentially exhibit structural flexibility, reliability through redundancy, and simple hardware as compared to a complex individual robot.

Example  The first robot rover exploration of Mars was in 1997—the Mars Pathfinder Mission. The rover, named Sojourner, looked like this:

You can see the whip antenna. The radio link was used to send commands from Earth to the rover and receive images and other data from the rover. Because the rover radio had a signal range similar
to a walkie-talkie, namely, about 10 m, all rover communication was done with the aid of the lander communications interface, like this:

![Mars Pathfinder Lander and Rover Telecommunications Links](image)

The rover telecommunications system was a two-way wireless UHF (Ultra High Frequency) radio link between the lander and the rover. The rover’s and lander’s UHF antennas worked very much like the antennas on walkie talkies or on car radios, using a “monopole” antenna. The signal to be transmitted enters the antenna through a coaxial connector located at the bottom, travels through a short section of balanced coaxial line and is radiated by the monopole.

It is desirable to have an antenna radiation pattern shaped to match its particular application. Satellite dishes are designed to look at a particular location in space and therefore need to have narrow and directive radiation patterns. The rover antenna did not need to look up into space, but rather needed to look horizontally in 360 degrees given that the lander could be in any direction. An ideal monopole has a 360 degree radiation pattern that is donut shaped, oriented horizontally. It is not meant to look straight up, and has poor reception in that direction. Certain metallic or rocky structures and ground reflections near the monopole antenna will distort its radiation pattern and cause holes or null zones to form. In these null zones the signal can drop significantly, causing poor reception. It is important to know where the rover is relative to the lander when these null zones exist, for if two nulls happened to get pointed at each other, there may be no radio reception at all.

As discussed, Sojourner had to be within about 10 m of the lander to send radio signals. This obviously is a limitation for scientific experiments. To get a longer radio link, one could use a higher power signal. But power on a Mars robot is a luxury. Another solution is an antenna array.

The purpose of an antenna array is to achieve directivity, the ability to send the transmitted signal in a preferred direction. If a large number of array elements can be used, it is possible to greatly enhance the strength of the signal transmitted in a given direction. An interesting example is the High Frequency Active Auroral Research Program (HAARP) in Alaska, whose purpose is to study the ionosphere. The site consists of a $15 \times 12$ array of dipole antennas:
Robot team

This leads us to the following scenario. A team of rovers doing scientific experiments. Each has, besides scientific instruments, a radio transceiver and an antenna. When it’s time to communicate with the lander, the rovers arrange themselves in a suitable formation to become an antenna array in order to optimize the signal strength. In general, the larger the array, the higher the resolution it can achieve.

The preceding example leads to the sub-question

Can we get a group of robot rovers, placed initially at random, to form a circle—or other shape?

We study this question for the simplest possible model of a robot—a point moving in the plane—and for a model of a wheeled rover.

1.2 A Bit of History

In 1987 Craig Reynolds introduced a model and wrote a program called boids that simulates a flock of birds in flight; they fly as a flock, with a common average heading, and they avoid colliding with each other. Each bird has a local control strategy—there’s no leader broadcasting instructions—yet a desirable overall group behaviour is achieved. The local strategy of each bird has three components:

1. separation—steer to avoid crowding;
2. alignment—steer towards the average heading of neighbours;
3. cohesion—steer towards the average position of neighbours.

Following Reynolds is the influential paper


Vicsek et al. propose a simple discrete-time model of $n$ autonomous agents i.e., points or particles all moving in the plane with the same speed but with different headings. Each agents heading is updated using a local rule based on the average of its own heading plus the headings of its neighbors.
Agent $i$’s neighbours at time $t$ are those agents that are either in or on a circle of pre-specified radius centred at agent $i$’s current position. The paper provides a variety of interesting simulation results that demonstrate that the nearest-neighbour rule can cause all agents eventually to move in the same direction despite the absence of centralized coordination and despite the fact that each agent’s set of nearest neighbours changes with time as the system evolves.

A second seminal paper is


Jadbabaie et al. study the Vicsek model and prove that all the heading angles converge to a common value provided the visibility graph of who can see whom is sufficiently connected.

Other motivation comes from biology: How do fish form and move in schools? How do birds flock? These are current questions in biology. A related area of research is biomimetics, that is, robotics inspired by biology.

**Suggestion** Google “emergent behavior.”

**Prerequisites**

1. Linear algebra.

2. Linear system theory; e.g., ECE411 or ECE557.

3. Helpful but not necessary: nonlinear system theory.

**1.3 References**

Chapter 2

The Point Robot

The course starts with the simplest possible robot, the point robot. We can then focus on coordinated behaviour without getting bogged down in vehicle dynamics.

2.1 The Model and a Control Problem

The point robot acts in continuous time, moves in the $x,y$-plane, and is linear and time-invariant (LTI):

Assume the velocities are controllable:

\[
\begin{align*}
\dot{x} &= v \\
\dot{y} &= w.
\end{align*}
\]

In vector form:

\[
\begin{align*}
\mathbf{q} &= \begin{bmatrix} x \\ y \end{bmatrix}, \\
\mathbf{u} &= \begin{bmatrix} v \\ w \end{bmatrix}, \\
\dot{\mathbf{q}} &= \mathbf{u}.
\end{align*}
\]

Equivalently, in complex variable form:

\[
q = x + jy, \quad u = v + jw, \quad \dot{q} = u.
\]
Notation

In this course we shall write a vector as an ordered tuple

\[ \mathbf{q} = (x, y) \]

or as a column vector

\[ \mathbf{q} = \begin{bmatrix} x \\ y \end{bmatrix}, \]

whichever is more convenient at the time. Likewise, we might associate

\[(u, v) \text{ and } \begin{bmatrix} u \\ v \end{bmatrix}, \]

where \( u \) is an \( m \)-tuple and \( v \) an \( n \)-tuple. (This permits us to avoid ugly expressions like \[ \begin{bmatrix} u^T \\ v^T \end{bmatrix}^T \]
for a column vector.)

Sample problem

The problem is to stabilize the robot to the origin. Example solution—move toward a beacon placed at the origin at a speed equal to half the current distance:

\[ \dot{\mathbf{q}} = u, \quad u = -q/2 \implies \dot{q} = -q/2. \]

This is linear time-invariant state feedback:

Suppose the robot has onboard an omni-directional camera, a computer, and a clock. Can this control law really be implemented? Assume there’s a planar world with a global coordinate system, \( \Sigma \). Assume there’s a beacon at the origin of \( \Sigma \) and that it is a cylinder. The robot’s coordinates in \( \Sigma \) are \((x, y)\). Assume the robot is programmed to know the diameter of the beacon, and can therefore compute its distance at any time. At \( t = 0 \) the robot begins to move in the direction of the centre of the beacon with speed \( d(t)/2 \), where \( d(t) \) is the distance from the robot to the beacon. This implements \( u = -q/2 \).

Now let’s discuss the distinction between global and local coordinates. Let \( \Sigma \) again be the global coordinate system. The robot is at location \( q \) wrt \( \Sigma \) and the beacon is at the origin wrt \( \Sigma \). On the
other hand, let $\Sigma_1$ be the local coordinate system on the robot. For example, the $x$-axis of $\Sigma_1$ might point toward the centre of the camera’s field of view. Assume $\theta_1$ is constant ($\Sigma_1$ doesn’t rotate).

With respect to $\Sigma_1$, the location of the beacon is $-e^{-j\theta_1}q$. So with respect to $\Sigma_1$, the previously proposed control law (move toward the beacon at a speed equal to half the distance) is $1u = -e^{-j\theta_1}q/2$. Relative to $\Sigma$, the direction $1u$ transforms to $u = e^{j\theta_1}1u = -q/2$. Thus relative to $\Sigma$ the overall system is $\dot{q} = -q/2$, as before, and convergence ensues.

### 2.2 Exercises

1. Suppose the beacon is a thin pole. The robot can see the beacon, but can’t compute its distance. Suppose at $t = 0$ the robot therefore moves at unit speed toward the beacon. Write the kinematic equation; write its solution.

2. This problem models the robot as having mass. Assume the forces are controllable:

   \[
   \begin{align*}
   M\ddot{x} &= f_x \\
   M\ddot{y} &= f_y.
   \end{align*}
   \]

   Study the problem of stabilizing to the origin. Discuss feasibility using an onboard camera.

3. Consider a helicopter (point mass in gravity field). Write the dynamic model and stabilize about some elevation.
Discuss feasibility using an onboard camera.
Chapter 3

Point Robot Formations

In this chapter we study some formation problems for the simplest robot, the kinematic point. The control laws are distributed, in that no one robot plays leader. This is in contrast with much early work (1970s) on decentralized control.

3.1 Cyclic Pursuit

3.1.1 Pursuit


Here $A$ moves at unit speed and $P$ pursues $A$ at unit speed. Pursuit means the velocity vector always points at $A$:

$$\dot{P} = \frac{A - P}{\|A - P\|}.$$  

The trajectory of $P$ is a pursuit curve. We extend this idea to robots.

Example four robots in cyclic pursuit
Model the robots as points in the complex plane: $z_1, \ldots, z_4 \in \mathbb{C}$. Take the local strategy to be cyclic pursuit, e.g., $z_1$ heads for $z_2$:

\[ \dot{z}_1 = z_2 - z_1 \]

Note that $|\dot{z}_1|$ is not constant as it was for Bouguer. The four equations are:

\[
\begin{align*}
\dot{z}_1 &= z_2 - z_1 \\
\dot{z}_2 &= z_3 - z_2 \\
\dot{z}_3 &= z_4 - z_3 \\
\dot{z}_4 &= z_1 - z_4
\end{align*}
\]

Here are two simulations (more than 4 robots) to illustrate the behaviour:

We observe that

- The centroid is stationary.
- Each robot converges to the centroid.
- The trajectories sometimes intersect, sometimes don’t; depends on initial configuration.

Furthermore
• Control is distributed: identical local strategies.
• Sensor requirements minimal: \( n \) information-flow links for \( n \) agents.
• There is an emergent behaviour: convergence to a common point.

Let’s prove the behaviour properties.

**Proof** First, the centroid is stationary:

\[
\begin{align*}
\dot{z}_1 &= z_2 - z_1 \\
\dot{z}_2 &= z_3 - z_2 \\
\dot{z}_3 &= z_4 - z_3 \\
\dot{z}_4 &= z_1 - z_4
\end{align*}
\]

Add:

\[
\dot{z}_1 + \dot{z}_2 + \dot{z}_3 + \dot{z}_4 = 0.
\]

Thus

\[
z_1(t) + \cdots + z_4(t) = \text{constant}.
\]

But

\[
\text{centroid} = \frac{z_1(t) + \cdots + z_4(t)}{4}.
\]

To prove convergence to the centroid, put the equations in vector form:

\[
\dot{z} = Mz,
\begin{bmatrix}
-1 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & -1 & 1 \\
1 & 0 & 0 & -1
\end{bmatrix}
, \quad z \in \mathbb{C}^4.
\]

Let \( \mathbf{1} = (1, 1, 1, 1) \) be the vector of 1s. Every row of \( M \) sums to zero, so 0 is an eigenvalue of \( M \) and \( \mathbf{1} \) is a corresponding eigenvector. Thus \( \text{span}\{\mathbf{1}\} \) is an invariant subspace, that is also the set of equilibria of the system \( \dot{z} = Mz \).

It remains to show that the other eigenvalues, the nonzero ones, are stable. We have

\[
M = A - I
\]

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{bmatrix}
--\text{a companion matrix}
\]

characteristic poly \( A = s^4 - 1 \).
So $M$ has one eig at $\lambda = 0$, the others being stable. □

We say the four robots **rendezvous**, i.e., they converge to a common point.\(^1\)

Rendezvous is a form of consensus, namely, the robots come to agree on a common point. A similar idea would be, if the robots were to converge to a circle of a prescribed radius, then they would have achieved distance consensus.

### 3.1.2 Rendezvous under Cyclic Pursuit

We extend the example to the general case. Consider $n > 1$ robots with the kinematic model

$$\dot{z}_i = u_i.$$  

Robot $i$ can measure the relative position of $i + 1$ (mod $n$)

$$y_i = z_{i+1} - z_i, \quad y_n = z_1 - z_n.$$  

The control law is to steer toward the next robot:

$$u_i = F_i y_i, \quad F_i = 1.$$  

A global behaviour emerges—rendezvous:

$$(\forall z(0) \in \mathbb{C}^n)(\exists z_{ss} \in \mathbb{C}) \lim_{t \to \infty} z(t) = z_{ss}1.$$  

This follows from the following basic fact:

**Lemma 1** Consider $\dot{z} = Mz$. Assume $0$ is a simple eigenvalue of $M$, $1$ is a corresponding eigenvector, and all other eigenvalues have negative real part. Then the robots rendezvous.

An eigenvalue $\lambda$ is **simple** if it has multiplicity 1 as a root of the characteristic polynomial. Equivalently, the eigenspace $\ker(M - \lambda I)$ is 1-dimensional. (The *kernel*, or nullspace, of a matrix is the subspace of vectors that the matrix maps to the zero vector: $\ker M = \{ x : Mx = 0 \}$.)

The robots are modeled as living in the complex plane, and they could be collocated at any point. In the aggregate state-space, $\mathbb{C}^n$, the set of points corresponding to where the robots are

---

\(^1\)The dictionary definition of “to rendezvous” is to meet at an appointed place. But the robots do not have this capability, not having GPS. The best they can do is meet at *some* common location.
collocated is exactly the 1-dimensional subspace spanned by 1. So the picture going with the lemma is that \( \ker M \) is the 1-dimensional subspace spanned by 1 and every trajectory of \( \dot{z} = Mz \) converges to \( \ker M \):

\[
\begin{array}{c}
\text{ker } M \\
\end{array}
\]

**Discussion**

You go to the zoo with a friend. At some time you unfortunately become separated. How can you meet up again? This is a kind of rendezvous problem. If you had pre-arranged that in the eventuality of becoming separated you would both go to the entrance (or some other beacon), there would be no problem, so we exclude this situation. Likewise, we exclude the possibility of making an announcement over a PA system. The real problem is to devise identical search procedures for you and your friend to guarantee meeting, preferably in a reasonable time.

Another example (after Naomi Leonard): \( n \) robot submarine rovers have been exploring the ocean; at the end of a certain time period it’s desired that they assemble at a common point so that their power supplies can be recharged. This is a rendezvous problem.

**Implementation details**

1. Assume each robot has onboard a camera, a computer, and a clock. Also, each robot has a spherical head and the heads are all of a different colour. (Thus one robot can compute the distance to another using a camera by measuring the diameter of the latter’s sphere.)

2. Assume there’s a supervisor (a human or computer) who can occasionally download instructions to the robots. We think of this as a hybrid control setup: The supervisor at the discrete task level; the robots at the lower level.

3. Prior to the rendezvous task, the supervisor downloads the pursuit instructions to the robots, for example, \( \text{red} \) should pursue \( \text{green} \). The supervisor also downloads a time value \( t_0 \) at which the rendezvous task should begin.

4. Assume that at time \( t = 0 < t_0 \) the robots are dispersed in some fashion, say, by the supervisor.

5. At \( t = t_0 \) cyclic pursuit would begin and rendezvous would occur (asymptotically).
Global vs local coordinates

Shown are two robots, a global frame $\Sigma$, and two local frames, $\Sigma_1, \Sigma_2$, rotated by $\theta_1, \theta_2$, respectively. Wrts $\Sigma_1$, robot 1 sees robot 2 at the location $e^{-j\theta_1}(z_2 - z_1)$ and therefore applies the velocity input $u_1 = e^{-j\theta_1}(z_2 - z_1)$. So wrt $\Sigma$ we have as before

$$\dot{z}_1 = u_1 = e^{j\theta_1} u_1 = z_2 - z_1.$$  

Visibility graph

A key tool is the concept of visibility graph: The visibility graph for the cyclic pursuit setup is

Each node stands for a robot, and an arc from 1 to 2 means “1 can see (or sense) 2.” The adjacency
The matrix of the previous graph is

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{bmatrix}.
\]

That is, \( a_{ij} = 1 \) if there's an arrow from node \( i \) to node \( j \); otherwise, \( a_{ij} = 0 \). The **out-degree** of a node is the number of arrows leaving it, and \( D \) is the diagonal matrix of out-degrees; in this example \( D = I \). The **Laplacian** of the graph is \( L := D - A \). Thus the dynamic matrix \( M \) in the cyclic pursuit equation \( \dot{z} = Mz \) equals \(-L\). These concepts extend to general directed graphs.

**Target problem**

In cyclic pursuit, robot \( i \) is assumed to see robot \( i + 1 \) with perfect accuracy and no matter how far apart they are. Pretty idealized. Here’s a more realistic scenario:

1. There are \( n \) robots. At time \( t \), robot \( i \) can see (or otherwise sense) a subset \( \mathcal{N}_i(t) \) of the others. He heads for their centroid (or some other linear combination of their positions).

2. The set \( \mathcal{N}_i(t) \) may change with time, as for example if the sensor is a camera with a limited field of view, such as a cone. So the visibility graph is time-varying.

Can an emergent collective behaviour be assured? We’ll study this later in the course.

**Achievable formations**

Instead of convergence to a point, can we get three robots to form a triangle by a distributed strategy?

Let’s study this for a general formation and \( n \) robots.

We allow each robot to pursue a **virtual displacement** of the next robot:

\[
\dot{z}_i = (z_{i+1} + c_i) - z_i, \quad i = 1, \ldots, n - 1,
\]

\[
\dot{z}_n = (z_1 + c_n) - z_n.
\]

The vector form is

\[
\dot{z} = Mz + c. \tag{3.1}
\]
Let \( \mathbf{1} \) again denote the vector of all 1s: an eigenvector of \( M \) for the zero eigenvalue. Pre-multiply (3.9) by \( \mathbf{1}^T \):
\[
\frac{d(\mathbf{1}^Tz)}{dt} = \mathbf{1}^Tc.
\]
Thus if \( \mathbf{1}^Tc \neq 0 \), that is, if the centroid of the points \( c_1, \ldots, c_n \) is not at the origin, then the centroid of the robots moves off to infinity. To avoid this, we must assume that \( \mathbf{1}^Tc = 0 \). Then \( c \) lies in the stable eigenspace of \( M \).

**Theorem 1** Concerning the system \( \dot{z} = Mz + c \), assume the centroid of the points \( c_1, \ldots, c_n \) is at the origin. Let \( d \) denote the unique vector satisfying \( Md + c = 0 \), \( d \perp \mathbf{1} \). Then for every initial positions of the robots, the centroid of the points \( z_1(t), \ldots, z_n(t) \) is stationary and robot \( i \) converges to this centroid displaced by \( d_i \).

**Proof** The equation
\[
\dot{z} = Mz + c
\]
can be written as
\[
\frac{d}{dt}[z(t) - d] = M[z(t) - d + d] + c = M[z(t) - d].
\]
From our previous analysis (cyclic pursuit with no virtual displacements), the centroid of the points \( z_1(t) - d_1, \ldots, z_n(t) - d_n \) is stationary and every \( z_i(t) - d_i \) converges to this centroid.

But the centroid of the points \( z_1(t) - d_1, \ldots, z_n(t) - d_n \) equals the centroid of the points \( z_1(t), \ldots, z_n(t) \) since \( \mathbf{1} \perp d \).

**Simulations**

![Simulation Diagrams](image-url)

Triangle: \( c = \begin{bmatrix} -5 + j5\sqrt{3} \\ -5 + j5\sqrt{3} \\ 10 \\ 10 \\ -5 - j5\sqrt{3} \\ -5 - j5\sqrt{3} \end{bmatrix} \), Line: \( c = \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} \)

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3.1.3 Formation Evolution

We continue with robots under cyclic pursuit. So far, robots are points. So collisions are rare! Still, it’s interesting to study the time evolution of formations. The case \( n = 2 \) is trivial—the two robots move in a straight line toward each other. If four robots are initially in a square, they remain so (an exercise). More generally, if the robots are initially arranged in a convex polygon formation, they remain so. We intend now to extend this result.

We say that \( n \) points are in a **counterclockwise star formation** if they are arranged like this:

![Diagram of a counterclockwise star formation](image)

that is, all \( r_i > 0 \), all \( \alpha_i > 0 \), and \( \sum_{i=1}^{n} \alpha_i = 2\pi \). Here are some examples:

- convex polygon
- star formation

Obviously, a convex polygon is a special case of a star formation. Similarly for **clockwise star formation**. We consider only counterclockwise star formations, since clockwise star formations can be treated analogously.

**Theorem 2** Suppose that \( n \) (\( > 2 \)) distinct points initially are arranged in a counterclockwise star formation. Under cyclic pursuit the points remain in a counterclockwise star formation. (In particular, they never collide.)
Let’s prove the theorem when \( n = 3 \), which is not too hard. Then the theorem says: If the three points initially form a triangle, then they always do, that is, the triangle never collapses to a line or point.

**Proof**  Assume the initial setup is

\[
\begin{align*}
\text{Define } F &= \Im\left((z_1 - z_2)(z_3 - z_2)\right) \\
&= \Im\left(r_1 r_3 e^{j(\theta_3 - \theta_1)}\right) = r_1 r_3 \sin(\alpha).
\end{align*}
\]

Since the points move, \( \alpha \) and \( F \) are functions of \( t \). By assumption, \( 0 < \alpha(0) < \pi \), and so \( F(0) > 0 \).
We have

\[
\dot{F} = \frac{d}{dt} \Re\left\{ (z_1 - z_0)(z_2 - z_0) \right\}
\]

\[
= \Re\left\{ (\dot{z}_1 - \dot{z}_2)(z_3 - z_2) \right\} + \Re\left\{ (z_1 - \dot{z}_2)(\dot{z}_3 - \dot{z}_2) \right\}
\]

\[
= \Re\left\{ (\dot{z}_2 - z_1 - z_3 + z_0)(z_3 - z_2) \right\} + \Re\left\{ (z_1 - z_2)(z_1 - z_3 - z_0 + z_0) \right\}
\]

\[
= -\Re\left\{ (z_1 - z_2)(z_3 - z_0) \right\} - \Re\left\{ (z_2 - z_1 - z_3)(z_0 - z_2) \right\}
\]

\[
= -F - 0 + \Re\left\{ (z_1 - z_2)(z_1 - z_3) \right\} - F
\]

\[
= -2F + \Re\left\{ (z_1 - z_2)(z_1 - z_2 + z_2 - z_3) \right\}
\]

\[
= -2F + \Re\left\{ (z_1 - z_2)(z_1 - z_2) \right\} - \Re\left\{ (z_1 - z_2)(z_3 - z_2) \right\}
\]

\[
= -3F.
\]

Thus \( F(t) = e^{-3t}F(0) > 0 \) for all \( t > 0 \), and so \( 0 < \alpha(t) < \pi \) for all \( t > 0 \). So the triangle never collapses to a line. \( \square \)

The proof of the theorem for \( n > 3 \) requires some machinery. We begin with a tool for studying angles. Consider the setup:

![Diagram](image)

Lemma 2 Define \( F = \Re\{(z_1 - z_0)(z_2 - z_0)\} \). Then

(a) \( 0 < \alpha < \pi, \ r_1 > 0, \) and \( r_2 > 0 \) iff \( F > 0 \),

(b) \( \pi < \alpha < 2\pi, \ r_1 > 0, \) and \( r_2 > 0 \) iff \( F < 0 \),

(c) the points are collinear iff \( F = 0 \).

Proof Polar form:

\( z_1 - z_0 = r_1 e^{i\theta_1}, \quad z_2 - z_0 = r_2 e^{i\theta_2}. \)

Then

\[
F = \Re\{(z_1 - z_0)(z_2 - z_0)\}
\]

\[
= \Re\{r_1 e^{-i\theta_1} r_2 e^{i\theta_2}\}
\]

\[
= \Re\{r_1 r_2 e^{i\alpha}\}
\]

\[
= r_1 r_2 \sin(\alpha).
\]
Thus, $r_1 > 0$, $r_2 > 0$, and $0 < \alpha < \pi$ iff $F > 0$; and $r_1 > 0$, $r_2 > 0$ and $\pi < \alpha < 2\pi$ iff $F < 0$. Also, $F = 0$ iff $\alpha = 0$, $\alpha = \pi$, $r_1 = 0$, or $r_2 = 0$, i.e., the points are collinear. □

**Lemma 3** Suppose that $n$ ($> 2$) points $z_1, \ldots, z_n$ form a counterclockwise star formation. Then $\alpha_i < \pi$.

**Proof** Suppose, e.g., $\alpha_1 \geq \pi$. Fix a coordinate system centered at $z_c$ with the positive real axis given by the ray from $z_c$ passing through $z_1$. Then the setup is

![Diagram](attachment:image.png)

But this is impossible. □

**Lemma 4** If $z_1, \ldots, z_n$ are collinear at some time, they are collinear for all time.

**Proof** Suppose the points are collinear at $t = t_1$. Reorient the coordinate system so that the points lie on the real axis $\mathbb{R}$ at $t = t_1$. Considering the system $\dot{z} = Mz$, $\mathbb{R}^n$ is an invariant subspace. Hence, $z_i(t) \in \mathbb{R}$ for all time, implying the points are collinear for all $t$. □

**Proof of Theorem 2** To simplify notation, we’ll do the case $n = 4$. The setup is

![Diagram](attachment:image.png)

Consider the functions

\[
F_1(t) = \Im\{(z_1(t) - z_c)(z_2(t) - z_c)\}
\]
\[
F_2(t) = \Im\{(z_2(t) - z_c)(z_3(t) - z_c)\}
\]
\[
F_3(t) = \Im\{(z_3(t) - z_c)(z_4(t) - z_c)\}
\]
\[
F_4(t) = \Im\{(z_4(t) - z_c)(z_1(t) - z_c)\}.
\]
By the definition of a counterclockwise star formation and Lemma 3, \( r_i(0) > 0 \) and \( 0 < \alpha_i(0) < \pi \) (recall that \( r_i = |z_i - z_c| \)). Hence, by Lemma 2, \( F_i(0) > 0 \), \( \forall i \). Likewise, we need to show that \( F_i(t) > 0 \) for all \( t \), implying \( r_i(t) > 0 \) and \( 0 < \alpha_i(t) < \pi \) for all \( t \).

For a proof by contradiction, suppose that some \( F_i \), in particular \( F_1 \), becomes zero and it does so for the first time at \( t = t_1 \). Not all \( F_i \)'s are zero at \( t_1 \), for then the points would be collinear, by Lemma 2, which would be a contradiction, by Lemma 4. Suppose in fact that \( F_2(t_1) > 0 \). We therefore have

\[
\begin{align*}
F_i(t) &> 0, \quad t \in [0, t_1), \ i = 1, \ldots, 4 \\
F_1(t_1) &= 0 \\
F_2(t_1) &> 0.
\end{align*}
\]

Taking the derivative along trajectories of the system \( \dot{z} = Mz \) and noting that \( \dot{z}_c = 0 \) (the centroid is stationary), we have

\[
\dot{F}_1 = \frac{d}{dt} (\Re\{z_1 - z_c\}(z_2 - z_c))
= \Re\{\overline{z}_1(z_2 - z_c)\} + \Re\{\overline{z}_1 - z_c\}\dot{z}_2
= \Re\{\overline{z}_2 - z_1\}(z_2 - z_c)\} + \Re\{\overline{z}_1 - z_c\}(z_3 - z_2)\}
= -\Re\{\overline{z}_1 - z_c\}(z_2 - z_c)\} + \Re\{\overline{z}_1 - z_c\}(z_3 - z_2)\}
= -\Re\{\overline{z}_1 - z_c\}(z_2 - z_c)\} + \Re\{\overline{z}_2 - z_c\}(z_2 - z_c)\}
= -\Re\{\overline{z}_1 - z_c\}(z_3 - z_c)\} - \Re\{\overline{z}_1 - z_c\}(z_2 - z_c)\}
= -F_1 + 0 + G_1 - F_1
= -2F_1 + G_1,
\]

where \( G_1 := \Re\{\overline{z}_1 - z_c\}(z_3 - z_c)\}. \) Observe that

\[
G_1 = r_1 r_3 \sin(\alpha_1 + \alpha_2).
\]

By the formula \( F_1 = r_1 r_2 \sin(\alpha_1) \), the condition \( F_1(t_1) = 0 \) implies that at \( t = t_1 \) at least one of the following holds:

1. \( r_1 = 0 \)
2. \( r_2 = 0 \)
3. \( \alpha_1 = 0 \) and \( r_1, r_2 > 0 \)
4. \( \alpha_1 = \pi \) and \( r_1, r_2 > 0 \).

Let us show that each of these is impossible:

2: \( r_2(t_1) = 0 \) is impossible since \( F_2(t_1) > 0 \).

4: Suppose \( \alpha_1(t_1) = \pi \) and \( r_1(t_1), r_2(t_1) > 0 \). Then the four points are collinear, which contradicts Lemma 4.
3. Suppose $\alpha_1(t_1) = 0$ and $r_1(t_1), r_2(t_1) > 0$. Then

$$G_1(t_1) = r_1(t_1)r_3(t_1)\sin[\alpha_1(t_1) + \alpha_2(t_1)]$$

$$= r_1(t_1)r_3(t_1)\sin[\alpha_2(t_1)]$$

$$= \frac{r_1(t_1)}{r_2(t_1)} F_2(t_1).$$

Hence $G_1(t_1) > 0$. By continuity, there exists $t_0$, $0 \leq t_0 < t_1$, such that $G_1(t) > 0$ for all $t \in [t_0, t_1]$. Also, by assumption, $F_1(t) > 0$ for $t \in [0, t_1)$. Hence,

$$\dot{F}_1 = -2F_1 + G_1 > -2F_1, \quad t \in [t_0, t_1).$$

Therefore

$$F_1(t) > e^{-2(t-t_0)}F_1(t_0) > 0, \quad t \in [t_0, t_1).$$

By continuity of $F_1$, $F_1(t_1) > 0$, a contradiction.

1. Finally, suppose $r_1(t_1) = 0$, that is, $z_1(t_1)$ and $z_c$ coincide. The configuration must look like this at $t = t_1$ (the picture is rotated to put $z_2$ on the negative real axis):

It follows that

$$\dot{z}_1(t_1) = z_2(t_1) - z_1(t_1) = z_2(t_1) < 0,$$

that is, $z_1$ is moving horizontally and to the left at $t_1$. So by continuity, just prior to $t_1$, the picture must have been

But this is not a counterclockwise star formation, contradiction.
3.1.4 Exercises

1. Four robots; $z_1$ heads for the centroid of the others:

$$
\dot{z}_1 = \frac{1}{3}(z_2 + z_3 + z_4) - z_1.
$$

And so on.

\[\text{Draw the visibility graph. Write down } A, D, L, M. \text{ Study emergent behaviour.}\]

2. For 4 robots in cyclic pursuit, show that there is a unique $d$ in the stable eigenspace such that $Md + c = 0$.

3. Prove that if 4 robots are initially arranged in a square, they remain in a square under cyclic pursuit.

4. It can be proved that when point robots move according to the cyclic pursuit control law, they converge to an elliptical point. This means that, although they are converging to their starting centroid, if you zoom in at the right speed, you will see them converge to motion around a stationary ellipse. This exercise is to see this phenomenon in simulation. Write a program to do this:

(a) Take $n = 6$ point robots in cyclic pursuit, $\dot{z}_i = z_{i+1} - z_i$. Form the matrix $M$ so that $\dot{z} = Mz$.

(b) Compute the eigenvalues of $M$. They lie on a circle centred at $-1$, radius 1. Let $-\gamma$ denote the real part of rightmost nonzero eigenvalues ($\gamma = 0.5$).

(c) Randomly generate an initial complex vector $z(0)$. Translate it so that the centroid of its components is zero.

(d) Simulate $\dot{z} = Mz$.

(e) Zoom in by defining $w(t) = e^{\gamma t}z(t)$. (Or by simulating $\dot{w} = (M + \gamma I)w$.)

(f) Plot the portraits in the complex plane of $w_1(t), \ldots, w_6(t)$.

5. Three robots: Can a triangular formation be achieved using only onboard sensors?

6. Here’s a quasi-distributed algorithm to get $n$ robots to line up, equally spaced apart, with $z_1$ and $z_n$ forming the ends of the line:

(a) Let $z_1$ not move: $\dot{z}_1 = 0$. 

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Let $z_2$ move toward the centroid of the “local” subset $\{z_1, z_2, z_3\}$:
\[ \dot{z}_2 = \frac{1}{3}(z_1 + z_2 + z_3) - z_2. \]

Let $z_3$ move toward the centroid of the “local” subset $\{z_2, z_3, z_4\}$:
\[ \dot{z}_3 = \frac{1}{3}(z_2 + z_3 + z_4) - z_3. \]

Etc.

Let $z_{n-1}$ move toward the centroid of the “local” subset $\{z_{n-2}, z_{n-1}, z_n\}$:
\[ \dot{z}_{n-1} = \frac{1}{3}(z_{n-2} + z_{n-1} + z_n) - z_{n-1}. \]

Let $z_n$ not move: $\dot{z}_n = 0$.

Prove for, say, $n = 5$, that the robots converge to become uniformly spaced on a line.

### 3.1.5 References


3.2 Nonnegative Matrices

Nonnegative matrices arise in the study of graphs (adjacency matrices). We need them for visibility graphs. The main result is the Perron-Frobenius theorem. In this section all matrices are real and square.

3.2.1 Example

We begin with an example to illustrate the theorem. Consider \( n \) cities. Let \( p_i(k) \) denote the population of city \( i \) on day \( k \). Assume each day a fraction, \( a_{ij} \), of people move from city \( j \) to city \( i \); the fraction \( a_{jj} \) remain.

Then

\[
p_i(k + 1) = a_{i1}p_1(k) + \cdots + a_{in}p_n(k),
\]

or in vector form

\[
p(k + 1) = Ap(k).
\]

We ask the question: What’s the limiting population distribution, \( \lim_{k \to \infty} p(k) \)? Since \( p(k) = A^kp(0) \), the question involves \( \lim_{k \to \infty} A^k \).

Note that \( 0 \leq a_{ij} \leq 1 \) (each element is a fraction), and each column-sum equals 1 (every person in city \( j \) goes somewhere, or stays put). Such \( A \) is called a column stochastic matrix since each column has the properties of a discrete probability density function.

Some terminology: The spectrum is the set of all eigenvalues. Notation \( \sigma(A) \). Also, the spectral radius is the maximum magnitude of all the eigenvalues:

\[
\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}
\]

Let’s specialize to two cities with

\[
A = \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix}.
\]

Defining \( \mathbf{1} = (1, 1) \), we see \( \mathbf{1}^TA = \mathbf{1}^T \), so 1 is an eigenvalue. We compute

\[
\lambda_1 = 1, \quad v_1 = \begin{bmatrix} 0.4 \\ 0.2 \end{bmatrix}; \quad \lambda_2 = 0.4, \quad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.
\]
So if 
\[ V = \begin{bmatrix} v_1 & v_2 \end{bmatrix}, \quad D = \text{diag}(\lambda_1, \lambda_2), \]
then 
\[ AV = VD, \quad A^k V = VD^k \rightarrow V \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} v_1 & 0 \end{bmatrix}. \]

Noting \( \mathbf{1} \perp v_2 \), we get 
\[
v_1 \mathbf{1}^T V = v_1 \mathbf{1}^T \begin{bmatrix} v_1 & v_2 \end{bmatrix} = v_1 \begin{bmatrix} 1^T v_1 & 0 \end{bmatrix} = (1^T v_1) v_1 \begin{bmatrix} 1 & 0 \end{bmatrix} = (1^T v_1) \begin{bmatrix} v_1 & 0 \end{bmatrix},
\]
so 
\[ \lim_k A^k V = \begin{bmatrix} v_1 & 0 \end{bmatrix} = \frac{1}{1^T v_1} v_1 \mathbf{1}^T V, \]
and hence 
\[ \lim_k A^k = \frac{1}{1^T v_1} v_1 \mathbf{1}^T. \]

Then 
\[ \lim_k p(k) = \frac{1^T p(0)}{1^T v_1} v_1 = (\text{total initial pop.}) \times \begin{bmatrix} 2/3 & 1/3 \end{bmatrix}, \]
i.e., the limiting population distribution is independent of the initial distribution.

**Recap of example**

1. 
\[ A = \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix} \]
2. Eigenvalues \( \lambda_1 = 1, \lambda_2 = 0.4 \). So \( \rho(A) \) is an eigenvalue. It’s simple (multiplicity 1).
3. Corresponding to \( \rho(A) \), \( A \) has a positive eigenvector, \( v_1 \).
4. \( \lim_k (\rho(A)^{-1} A)^k \) exists and is a rank 1 matrix of the form 
\[ \text{const.} \times v_1 \mathbf{1}^T. \]

Our goal is to generalize this example.

Intuitively why \( A \) has a real, positive eigenvalue and a positive eigenvector:
\[ A = \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix}. \]
The first quadrant is an invariant cone:

Doesn’t it seem plausible that for some $x \neq 0$, $x$ and $Ax$ are collinear?

### 3.2.2 Induced Matrix Norms

Consider a vector $x$ in $\mathbb{R}^n$. There are an infinite number of possible norms for $x$, the three most common being

$$
\|x\|_1 = \sum_i |x_i|, \quad \|x\|_2 = \left( \sum_i |x_i|^2 \right)^{1/2}, \quad \|x\|_\infty = \max_i |x_i|.
$$

Let $A$ be an $n \times n$ real matrix. Each of the above vector norms induces a norm on $A$:

$$
\|A\|_i = \max_{\|x\|_i = 1} \|Ax\|_i, \quad i = 1, 2, \infty.
$$

These are called **induced matrix norms**. Not all norms $\|A\|$ are induced by vector norms, e.g.,

$$
\max_{i,j} |a_{ij}|.
$$

**Theorem 3** For any induced matrix norm, $\rho(A) \leq \|A\|$ and $\rho(A) = \lim_{k \to \infty} \|A^k\|^{1/k}$.

**Proof** Let $\|x\|$ be any vector norm. Then

$$
\|A\| = \max \{ \|Ax\| : \|x\| = 1 \} \\
\geq \max \{ \|Ax\| : x \text{ is an eigenvector, } \|x\| = 1 \} \\
= \max \{ \|\lambda x\| : \lambda \text{ is an eigenvalue, } x \text{ is a corresponding eigenvector, } \|x\| = 1 \} \\
= \max \{ |\lambda| : \lambda \text{ is an eigenvalue} \} \\
= \rho(A).
$$

For any positive integer $k$, $\rho(A^k) \leq \|A^k\|$. But also $\rho(A^k) = \rho(A)^k$. Thus $\rho(A) \leq \|A^k\|^{1/k}$.

Let $\varepsilon > 0$ and define

$$
B = \frac{1}{\rho(A) + \varepsilon} A.
$$

Then $\rho(B) < 1$, so $B^k \to 0$ as $k \to \infty$. So for large enough $k$, $\|B^k\| < 1$, i.e.,

$$
\|A^k\| < (\rho(A) + \varepsilon)^k,
$$

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\[ \|A^k\|^{1/k} < \rho(A) + \varepsilon. \]

Thus we have, for large enough \( k \)
\[ \rho(A) \leq \|A^k\|^{1/k} < \rho(A) + \varepsilon. \]

Let \( k \to \infty \) then use that \( \varepsilon \) was arbitrary. \( \square \)

**Corollary 4**

\[ \rho(A) \leq \max_j \sum_i |a_{ij}| = \|A\|_1, \quad \rho(A) \leq \max_i \sum_j |a_{ij}| = \|A\|_\infty \]

### 3.2.3 Positive and Nonnegative Matrices

A matrix \( A \) is **nonnegative** if all \( a_{ij} \geq 0 \). This is written \( A \geq 0 \). We say \( A \) is **positive** if all \( a_{ij} > 0 \). Written \( A > 0 \). Extend this idea to \( A \geq B \ (A - B \geq 0) \) and \( A > B \ (A - B > 0) \). Likewise for vectors, e.g., \( x \geq y \) means \( x_i \geq y_i \) for every \( i \).

**Lemma 5** Let \( A \geq 0 \). If \( A \) has constant column sums, then \( \rho(A) = \|A\|_1 \). If \( A \) has constant row sums, then \( \rho(A) = \|A\|_\infty \).

**Proof** We know \( \rho(A) \leq \|A\|_\infty \). Suppose \( A \) has constant row sums and denote the constant by \( \delta \); thus \( \delta = \|A\|_\infty \). Then \( 1 \) satisfies \( A1 = \delta 1 \). It follows that \( \delta \leq \rho(A) \), hence \( \|A\|_\infty \leq \rho(A) \). For column sums, apply the same argument to \( A^T \). \( \square \)

**Lemma 6** If \( 0 \leq A \leq B \), then \( \rho(A) \leq \rho(B) \).

**Proof** Here's the proof with a few details left out:

\[
0 \leq A \leq B \quad \Rightarrow \quad 0 \leq A^2 \leq B^2 \\
\quad \Rightarrow \quad 0 \leq A^k \leq B^k \\
\quad \Rightarrow \quad \|A^k\|_1 \leq \|B^k\|_1 \\
\quad \Rightarrow \quad \|A^k\|_1^{1/k} \leq \|B^k\|_1^{1/k} \\
\quad \Rightarrow \quad \lim_{k} \|A^k\|_1^{1/k} \leq \lim_{k} \|B^k\|_1^{1/k} \\
\quad \Rightarrow \quad \rho(A) \leq \rho(B). \]

\( \square \)
Theorem 5 If $A \geq 0$, then
\[
\min_j \sum_i a_{ij} \leq \rho(A) \leq \max_j \sum_i a_{ij} = \|A\|_1
\]
and
\[
\min_i \sum_j a_{ij} \leq \rho(A) \leq \max_i \sum_j a_{ij} = \|A\|_\infty.
\]

Proof We already did the upper bounds. Let’s do
\[
\delta := \min_j \sum_i a_{ij} \leq \rho(A).
\]
To see more clearly, let’s specialize to $n = 2$. The idea is to scale $A$ so that it has constant column sums.

Let’s say the second column sum is the minimum of the two, i.e., $\delta$. Then we can scale down the first:
\[
\begin{bmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{bmatrix}
\begin{bmatrix}
  \alpha & 0 \\
  0 & 1
\end{bmatrix}
= \text{constant column sums } \delta.
\]
This has the form
\[
AB = \text{constant column sums } \delta.
\]
By Lemma 5, $\rho(AB) = \|AB\|_1 = \delta$. But $AB \leq A$ and so by Lemma 6, $\rho(AB) \leq \rho(A)$.

Corollary 6 If $A \geq 0$ and $x > 0$, then
\[
\min_j x_j \sum_i \frac{a_{ij}}{x_i} \leq \rho(A) \leq \max_j x_j \sum_i \frac{a_{ij}}{x_i}
\]
and
\[
\min_i \frac{1}{x_i} \sum_j a_{ij} x_j \leq \rho(A) \leq \max_i \frac{1}{x_i} \sum_j a_{ij} x_j.
\]

Proof Let $X = \text{diag}(x_1, \ldots, x_n)$. Apply the theorem to
\[
X^{-1}AX = \begin{bmatrix}
  \frac{x_1}{x_1} a_{11} & \frac{x_2}{x_1} a_{12} & \cdots \\
  \frac{x_1}{x_2} a_{21} & \frac{x_2}{x_2} a_{22} & \cdots \\
  \vdots & \vdots & \ddots
\end{bmatrix}
\]
to get
\[
\min_j x_j \sum_i \frac{a_{ij}}{x_i} \leq \rho(X^{-1}AX) \leq \max_j x_j \sum_i \frac{a_{ij}}{x_i}
\]
and
\[
\min_i \frac{1}{x_i} \sum_j a_{ij} x_j \leq \rho(X^{-1}AX) \leq \max_i \frac{1}{x_i} \sum_j a_{ij} x_j.
\]
But $\rho(X^{-1}AX) = \rho(A)$.
3.2.4 Perron’s Theory, 1907

Now we turn to positive matrices. New notation: \( |A| = [|a_{ij}|] \). Likewise |x|. A positive matrix has a nonzero eigenvalue, that is, a nilpotent matrix can’t be positive. Moreover, the spectral radius is an eigenvalue and it has a positive eigenvector:

**Theorem 7** If \( A > 0 \), then \( \rho(A) > 0 \), \( \rho(A) \) is an eigenvalue, and it has a positive eigenvector.

**Proof** That \( \rho(A) > 0 \) follows immediately from Theorem 5, i.e.,

\[
\min_j \sum_i a_{ij} \leq \rho(A).
\]

Now suppose \( \lambda \) is an eigenvalue of maximum magnitude and \( x \) is a corresponding eigenvector, that is, \( Ax = \lambda x \), \( x \neq 0 \), and \( |\lambda| = \rho(A) \). We’ll show \( A|x| = \rho(A)|x| \) and \( |x| > 0 \).

We have

\[
\rho(A)|x| = |\lambda||x| \quad \text{(given)}
\]

\[
= |\lambda x| \quad \text{(identity)}
\]

\[
= |Ax| \quad \text{(given)}
\]

\[
\leq |A||x| \quad \text{(identity)}
\]

\[
= A|x| \quad \text{(since } A > 0 \).
\]

Define \( y := A|x| - \rho(A)|x| \geq 0 \). We want to show that \( y = 0 \).

Suppose to the contrary that \( y \neq 0 \). Then

\[
y \geq 0 , y \neq 0 , A > 0 \implies Ay > 0 ,
\]

so

\[
0 < Ay = A(A|x| - \rho(A)|x|) = AA|x| - \rho(A)A|x|.
\]

Define \( z = A|x| \). Then \( Az > \rho(A)z \) and so

\[
(\forall i) \left\{ \rho(A)z_i < \sum_j a_{ij}z_j \right\} \implies \rho(A) < \min_i \frac{1}{z_i} \sum_j a_{ij}z_j
\]

\[
\implies \rho(A) < \rho(A) \text{ by Corollary 6}.
\]

Impossible. Thus \( y = 0 \).

Remains to show \( |x| > 0 \). Now

\[
x \neq 0 , |x| \geq 0 , A > 0 \implies A|x| > 0 .
\]

Thus

\[
0 = y = A|x| - \rho(A)|x| \implies |x| = \rho(A)^{-1}A|x| > 0 .
\]
Theorem 8 If $A > 0$, then $\rho(A)$ is the unique eigenvalue of maximum magnitude.

Proof Suppose $\lambda$ is an eigenvalue of maximum magnitude and $x$ is a corresponding eigenvector, that is, $Ax = \lambda x$, $x \neq 0$, and $|\lambda| = \rho(A)$. We’ll show that the eigenvector $x$ can be rotated to be positive, that is, we’ll show there’s an angle $\theta$ such that $e^{i\theta}x > 0$.

As we showed in the proof of Theorem 7, $A|x| = \rho(A)|x|$ and $|x| > 0$. Now we have, for each $i$,

$$
\rho(A)|x_i| = |\lambda||x_i| \quad \text{(since } |\lambda| = \rho(A)\text{)}
= |\lambda x_i|
= |\sum_k a_{ik}x_k| \quad \text{(since } Ax = \lambda x\text{)}
\leq \sum_k |a_{ik}x_k| \quad \text{(triangle inequality)}
= \sum_k a_{ik}|x_k| \quad \text{(since } A > 0\text{)}
= \rho(A)|x_i| \quad \text{(since } A|x| = \rho(A)|x|\text{)}.
$$

Thus $|\sum_k a_{ik}x_k| = \sum_k |a_{ik}x_k|$, equality in the triangle inequality. E.g., for $n = 2$:

Thus $\exists \theta$ such that $e^{j\theta}a_{ik}x_k > 0$, so $e^{j\theta}x_k > 0$. Then

$$Ax = \lambda x \implies Ae^{j\theta}x = \lambda e^{j\theta}x \implies A|x| = \lambda|x|.$$ 

Since $|x| > 0$, $\lambda$ must be real and positive; hence $\lambda = \rho(A)$.

So the eigenvalues of a positive matrix look like
Eigenvalue multiplicity

We now review the notions of algebraic and geometric multiplicity. Consider the matrices
\[
\begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}, \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}, \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}, \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Algebraic multiplicity refers to the multiplicity of an eigenvalue as a root of the characteristic polynomial. Geometric multiplicity refers to the dimension of the eigenspace, that is, the number of linearly independent eigenvectors. So for the examples we have

<table>
<thead>
<tr>
<th>example</th>
<th>eigenvalue</th>
<th>algebraic mult.</th>
<th>geometric mult.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Finally, an eigenvalue is **simple** if its algebraic multiplicity is 1.

Convergence:

**Theorem 9** Suppose \( A > 0 \) and let \( \rho = \rho(A) \). Let \( x \) be a positive eigenvector of \( A \) corresponding to \( \rho \) and let \( z \) be a positive eigenvector of \( A^T \) corresponding to \( \rho \). Define \( y = (x^Tz)^{-1}z \) and \( L = xy^T \). Then \( \lim_{k \to \infty} (\rho^{-1}A)^k = L \).

**Proof** Note that \( x^Tz \neq 0 \) since \( x > 0, z > 0 \). Since \( y = (x^Tz)^{-1}z \) it follows that \( x^Ty = 1 \), and so \( L^2 = L \). Likewise, by induction \( L^k = L \).

Next,
\[
AL = Ax y^T = \rho xy^T = \rho L
\]
and by induction $A^kL = \rho^kL$. Likewise $LA^k = \rho^kL$. Then

$$(A - \rho L)^2 = A^2 - \rho AL - \rho LA + \rho^2 L^2 = A^2 - \rho^2 L - \rho^2 L + \rho^2 L = A^2 - \rho^2 L,$$

and by induction

$$(A - \rho L)^k = A^k - \rho^k L$$

and hence

$$[\rho^{-1}(A - \rho L)]^k = (\rho^{-1}A)^k - L.$$  

It remains to show that $\lim_{k \to \infty}[\rho^{-1}(A - \rho L)]^k = 0$, equivalently, $\rho > \rho(A - \rho L)$.

Let $\lambda$ be a nonzero eigenvalue of $A - \rho L$ and $v$ a corresponding eigenvector. Then

$$(A - \rho L)v = \lambda v \implies L(A - \rho L)v = \lambda Lv$$
$$\implies LAv - \rho L^2 v = \lambda Lv$$
$$\implies \rho Lv - \rho Lv = \lambda Lv$$
$$\implies Lv = 0$$
$$\implies Av = \lambda v,$$

So $\lambda$ is an eigenvalue of $A$. Hence, either $|\lambda| < \rho$—in which case we’re done—or else $\lambda = \rho$. In the latter case, since the geometric multiplicity of $\rho$ as an eigenvalue of $A$ is 1, it must be that $v = x$ (or some scalar multiple). Then

$$(A - \rho L)x = \rho x \implies (A - \rho xy^T)x = \rho x$$
$$\implies Ax - \rho x = \rho x$$
$$\implies \rho x - \rho x = \rho x$$
$$\implies \rho = 0,$$

which is false. Thus $\lambda \neq \rho$. \hfill \Box

The eigenvalue of maximum magnitude is simple:

**Theorem 10** If $A > 0$, then $\rho(A)$ is a simple eigenvalue.

**Proof** We use the setup in Theorem 9. Since $L = xy^T$, $L$ has rank 1, and hence has only one nonzero eigenvalue. Since $Lx = x$, that eigenvalue is 1. Then since

$$\lim_{k \to \infty}(\rho(A)^{-1}A)^k = L$$

it follows that $\rho(A)$ is a simple eigenvalue of $A$. \hfill \Box

Summary—

**Theorem 11 (Perron’s Theorem)** If $A > 0$, then

1. $\rho(A) > 0$
2. $\rho(A)$ is a simple eigenvalue
3. all other eigenvalues have $|\lambda| < \rho(A)$
4. $\rho(A)$ has a positive eigenvector
5. Let $x$ be a positive eigenvector of $A$ corresponding to $\rho(A)$ and let $z$ be a positive eigenvector of $A^T$ corresponding to $\rho(A)$. Define $y = (x^Tz)^{-1}z$ and $L = xy^T$. Then $\lim_{k \to \infty}(\rho(A)^{-1}A)^k = L$.  

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3.2.5 Irreducible Matrices

We want to extend Perron’s theory to nonnegative matrices, $A \geq 0$. The results do not extend in general, but they do extend for irreducible nonnegative matrices.

First, consider the integers $\{1, \ldots, n\}$. A permutation of them is a re-ordering, for example

$\{1, 2, 3, 4\} \mapsto \{2, 4, 3, 1\}$.

If you take the identity matrix and permute its rows, the result is called a permutation matrix. For example, the preceding permutation gives

$$
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}.
$$

Notice that this is also the permutation

$\{1, 2, 3, 4\} \mapsto \{4, 1, 3, 2\}$

applied to the columns of $I$. So in general a permutation matrix is obtained by permuting either the rows or the columns of the identity matrix. Each row has precisely one 1, and each column has precisely one 1. Also, a permutation matrix is orthogonal: $P^T P = PP^T = I$.

If $P$ is a permutation matrix, the operation $A \mapsto PA$ amounts to permuting the rows of $A$. Then $PA \mapsto PAP^T$ amounts to doing the same permutation on the columns of $PA$. Of course we could have permuted the columns first:

$$
A \mapsto A^T \mapsto PAP^T.
$$

A matrix is reducible if either

1. $n = 1$ and $A = 0$, or

2. there is a permutation matrix $P$ such that $PAP^T$ is block upper triangular, that is, has the form

$$
\begin{bmatrix}
B & C \\
0 & D
\end{bmatrix},
$$

with $B, D$ square.

Otherwise it is irreducible. Clearly, a positive matrix is irreducible.

Examples

$$
\begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}, \begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix}, \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}.
$$

The first is reducible, it being already upper triangular. In the second, if we interchange the rows, then interchange the columns, we get

$$
\begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix} \mapsto \begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix} \mapsto \begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}.
$$
So the second is reducible. If we do the same operations in the third matrix, we return to the same matrix:

\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \mapsto \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \mapsto \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}.
\]

So the third matrix is irreducible.

There are only finitely many permutation matrices of a given order, so in principle it’s possible to check in a finite number of steps if a matrix is irreducible.

Now we turn to the important connection between irreducible matrices and graphs. Let \(A \geq 0\) be of size \(n \times n\). From it construct a directed graph (digraph), \(G(A)\), with \(n\) nodes, \(p_1, \ldots, p_n\), and a directed arc \(p_i \to p_j\) iff \(a_{ij} > 0\).

**Example**

\[
A = \begin{bmatrix}
0 & 2 & 1 \\
0 & 3 & 1 \\
1 & 0 & 0
\end{bmatrix}
\]

A **directed path** in a digraph is a finite sequence of directed arcs

\[p_{i_1} \to p_{i_2} \to \cdots \to p_{i_m}\]

The **length** of a directed path is the number of arcs. We say \(p_j\) is **reachable** from \(p_i\) if there’s a directed path from \(p_i\) to \(p_j\). A digraph is **strongly connected** if every node is reachable from every other node.

Let \(a_{ij}^{(m)}\) denote element \(i, j\) of the matrix \(A^m\), \(m \geq 1\):

\[
A^m = \begin{bmatrix}
 a_{11}^{(m)} & a_{12}^{(m)} & \cdots \\
 a_{21}^{(m)} & a_{22}^{(m)} & \cdots \\
 \vdots & \vdots & \ddots
\end{bmatrix}.
\]

**Lemma 7** Let \(A \geq 0\). There exists a directed path of length \(m\) in \(G(A)\) from \(p_i\) to \(p_j\) iff \(a_{ij}^{(m)} > 0\).

**Proof** The proof is by induction on \(m\). The assertion is trivial for the case \(m = 1\): There exists a directed arc from \(p_i\) to \(p_j\) iff \(a_{ij} > 0\). For \(m = 2\), note that

\[
a_{ij}^{(2)} = \sum_k a_{ik} a_{kj}.
\]

Thus \(a_{ij}^{(2)} > 0\) iff (\(\exists k\))\(a_{ik} > 0 \& a_{kj} > 0\), that is, there are directed arcs \(p_i \to p_k\) and \(p_k \to p_j\). These form a directed path of length 2.

Now one assumes the claim is true for \(m = q\) and proves it for \(m = q + 1\). I leave it to you. \(\square\)
Corollary 12 Let $A \geq 0$. Then $\mathcal{G}(A)$ is strongly connected iff

$$(\forall i \neq j)(\exists m, 1 \leq m \leq n - 1)a_{ij}^{(m)} > 0.$$ 

Proof (⇐) From the lemma. (⇒) Let $i \neq j$ be arbitrary. There exists a directed path from $p_i$ to $p_j$. Since there are $n$ nodes altogether, there is a directed path from $p_i$ to $p_j$ with fewer than $n$ directed arcs. □

Lemma 8 Let $A \geq 0$. Then $\mathcal{G}(A)$ is strongly connected iff $(I + A)^{n-1} > 0$.

Proof In general we have

$$(I + A)^{n-1} = I + c_1 A + c_2 A^2 + \cdots + c_{n-1} A^{n-1},$$

where the combinatorial coefficients are all positive. Thus $(I + A)^{n-1} > 0$ iff

$$(\forall i \neq j)(\exists m, 1 \leq m \leq n - 1)a_{ij}^{(m)} > 0.$$ 

So the lemma follows from Corollary 12. □

Main result, a graphical characterization of irreducibility:

Theorem 13 Let $A \geq 0$. Then $A$ is irreducible iff $\mathcal{G}(A)$ is strongly connected.

Proof (⇐) If $A$ is reducible, $\exists P$ such that $PAP^T = \tilde{A}$ and

$$\tilde{A} = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}.$$ 

Then $(I + \tilde{A})^{n-1}$ is block upper triangular, and hence is not positive. So neither is $(I + A)^{n-1}$. So $\mathcal{G}(A)$ is not strongly connected, by Lemma 8.

(⇒) Assume $\mathcal{G}(A)$ is not strongly connected. By Lemma 8, some element of $(I + A)^{n-1}$, say $(i, j)$, is zero. By Lemma 7 $p_j$ is not reachable from $p_i$. Define

$$\mathcal{N}_1 = \{p_j\} \cup \{p_i : p_j \text{ is reachable from } p_i\}$$

and let $\mathcal{N}_2$ denote all the other nodes. These sets are both nonempty, e.g., $p_i \in \mathcal{N}_2$.

There are no directed paths from $\mathcal{N}_2$ to $\mathcal{N}_1$. To see this, suppose to the contrary that there’s a directed path from $p_k \in \mathcal{N}_2$ to $p_i \in \mathcal{N}_1$: \[\begin{array}{c}
\mathcal{N}_1 \\
\bullet p_i \\
\bullet p_j \\
\mathcal{N}_2 \\
\bullet p_k
\end{array}\]
Then $p_j$ is reachable from $p_k$, so $p_k \in N_1$, which is not true. By re-ordering the nodes according to the partition $N_1 \cup N_2$, we have the corresponding partition

$$PAP^T = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}.$$ 

Thus $A$ is reducible. □

3.2.6 Frobenius’ Theory, 1908-12

**Theorem 14 (Frobenius’ Theorem)** If $A \geq 0$ and $A$ is irreducible, then

1. $\rho(A) > 0$
2. $\rho(A)$ is a simple eigenvalue
3. $\rho(A)$ has a positive eigenvector

Proof omitted (similar to what we’ve already done).

3.2.7 Exercises

1. In $\mathbb{R}^2$ draw the unit balls

   $$\{ x : \| x \|_i \leq 1 \}, \quad i = 1, 2, \infty.$$

2. Prove that $\| A \|_1$ is the maximum absolute column sum,

   $$\| A \|_1 = \max_j \sum_i |a_{ij}|,$$

   and $\| A \|_\infty$ is the maximum absolute row sum,

   $$\| A \|_\infty = \max_i \sum_j |a_{ij}|.$$

3. Give an example of a matrix $A$ where $A \geq 0$, $A$ is reducible, and $\rho(A)$ is a simple eigenvalue of $A$.

4. Let

   $$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$ 

   Compute $\| A \|_1$, $\| A \|_\infty$, and $\rho(A)$.

5. Given an adjacency matrix, how could you test if it is irreducible?

   Give an example of a matrix $A$ where $A \geq 0$, $A$ is reducible, and $\rho(A)$ is a simple eigenvalue of $A$. 43
3.2.8 References


3.3 The Rendezvous Problem: Time-invariant Visibility Digraph

This section extends our study of the rendezvous problem via cyclic pursuit. We’re interested more generally in the question of what connectivity is required in the visibility graph.

3.3.1 Problem Statement

Let there be \( n \) robots:

\[
\dot{z}_i = u_i, \quad i = 1, \ldots, n
\]

Each can sense some neighbours using an onboard sensor. Let \( \mathcal{N}_i \) denote the sensed neighbours of robot \( i \). (Actual geometrical scope not relevant.) Assumption: \( \mathcal{N}_i \) not time-varying.

Example

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}
\]

\[
\mathcal{N}_1 = \{2\}, \quad \mathcal{N}_2 = \{3\}, \quad \mathcal{N}_3 = \{2, 4\}, \quad \mathcal{N}_4 = \{3\}
\]

Sensed variables are relative displacements since the sensors are onboard:

\[
y_1 = z_2 - z_1, \quad y_2 = z_3 - z_2, \quad y_3 = \begin{bmatrix} z_2 - z_3 \\ z_4 - z_3 \end{bmatrix}, \quad y_4 = z_3 - z_4.
\]

Allowed decentralized controllers:

\[
u_1 = F_1y_1, \quad u_2 = F_2y_2, \quad u_3 = F_3y_3, \quad u_4 = F_4y_4.
\]

Note the important property of this decentralized control structure: If \( y_1 = 0 \), that is, if \( z_1 \) is collocated with the only agent it senses, then \( u_1 = 0 \), so \( \dot{z}_1 = 0 \) and robot 1 doesn’t move. The problem is to find control gains \( F_I \) so that the robots rendezvous.

Note that the decentralized structure is shown by the block diagram

This is a bit misleading because the concept of a “sensor network” doesn’t bring out the point that the sensors are all local. It’s a network of mobile sensors.
So in general we have \( n > 1 \) robots:

\[
\dot{z}_i = u_i, \quad i = 1, \ldots, n
\]

Robot \( i \) senses the subset \( \mathcal{N}_i \). Let \( m_i \) denote the number of elements in \( \mathcal{N}_i \). Sensed variables: \( y_i \) is the vector with components \( z_j - z_i, \ j \in \mathcal{N}_i \). Thus \( \dim y_i = m_i \). Allowed controllers: \( u_i = F_i y_i \). So \( F_i \) is a real matrix of size \( 1 \times m_i \). The goal is **rendezvous**: For every set of initial positions, \( z_i(0) \), there exists a point \( z_{ss} \) such that every \( z_i(t) \) converges to \( z_{ss} \).

As we saw with cyclic pursuit, we have to allow \( z_{ss} \) to depend on the initial positions for the goal to be feasible. For example, if all the robots are initially placed at a point \( z_i(0) = w \), they’ll stay there forever. So it would necessarily follow that \( z_{ss} = w \).

### 3.3.2 Solution

Let \( G \) denote the visibility digraph for this problem. There are \( n \) nodes, \( p_i, i = 1, \ldots, n \), with an arc from \( p_i \) to \( p_j \) iff \( j \in \mathcal{N}_i \).

**Theorem 15**

1. The rendezvous problem is solvable iff \( G \) has a globally reachable node.

2. When the problem is solvable, one solution is

\[
F_i = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \quad \text{(row vector of 1s)}.
\]

**Example (Cont’d)**

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}
\]

Nodes \( \{2, 3, 4\} \) are globally reachable, so the problem is solvable. We had \( n = 4 \) with

\[
y_1 = z_2 - z_1, \quad y_2 = z_3 - z_2, \quad y_3 = \begin{bmatrix} z_2 - z_3 \\ z_4 - z_3 \end{bmatrix}, \quad y_4 = z_3 - z_4.
\]

Take

\[
F_1 = 1, \quad F_2 = 1, \quad F_3 = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad F_4 = 1.
\]

The closed-loop equations are

\[
\begin{align*}
\dot{z}_1 &= z_2 - z_1 \\
\dot{z}_2 &= z_3 - z_2 \\
\dot{z}_3 &= z_2 - z_3 + z_4 - z_3 \\
\dot{z}_4 &= z_3 - z_4.
\end{align*}
\]

Thus \( \dot{z} = Mz \), where

\[
M = \begin{bmatrix}
-1 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 \\
0 & 1 & -2 & 1 \\
0 & 0 & 1 & -1
\end{bmatrix}.
\]
eigs $M : \{-3, -1, -1, 0\}$
eigenvector for 0: all 1s.

Note that

$$
D = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}
$$

$$
L = D - A = \begin{bmatrix}
1 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 1
\end{bmatrix}, \quad M = -L
$$

---

3.3.3 Theory for the Proof

Consider a digraph. A nonempty set $\mathcal{U}$ of nodes is **closed** if every path starting in $\mathcal{U}$ stays in $\mathcal{U}$.

E.g., $\{3, 4\}$ is closed:

![Diagram of a digraph with nodes 1, 2, 3, and 4, and edges connecting them]

**Lemma 9** Assume $G$ has at least 2 nodes. It has a globally reachable node iff it does not have two disjoint closed subsets of nodes.

**Proof** ($\implies$) Let $\mathcal{V}$ denote the set of all nodes. Assume $\exists \mathcal{U}_1, \mathcal{U}_2$ disjoint, closed:

![Diagram of a digraph with $\mathcal{V}$ as the set of all nodes, $\mathcal{U}_1$ and $\mathcal{U}_2$ as disjoint subsets, and the union of $\mathcal{U}_1$ and $\mathcal{U}_2$ is closed]
Then there is no globally reachable node.

(⇐) Assume there is no globally reachable node. We shall construct two disjoint closed subsets of nodes. We illustrate with an example:

![Graph Diagram](https://example.com/graph.png)

Select any node, say \( p_{i_1} = 1 \), and partition the set of nodes as

\[
\mathcal{V} = \{p_{i_1}\} \cup \mathcal{V}_1 \cup \mathcal{V}_1' = \{1\} \cup \{3, 5, 7\} \cup \{2, 4, 6, 8\}
\]

where every node in \( \mathcal{V}_1 \) can reach \( p_{i_1} \) and no node in \( \mathcal{V}_1' \) can reach \( p_{i_1} \). Then \( \mathcal{V}_1' \) is closed, and nonempty since \( p_{i_1} \) is not globally reachable.

Next, select any node in \( \mathcal{V}_1' \), say \( p_{i_2} = 2 \). Notice that \( p_{i_2} \) is not globally reachable in the local subgraph with nodes \( \mathcal{V}_1' \). Thus we can partition \( \mathcal{V}_1' \) as

\[
\mathcal{V}_1' = \{p_{i_2}\} \cup \mathcal{V}_2 \cup \mathcal{V}_2' = \{2\} \cup \{8\} \cup \{4, 6\},
\]

where every node in \( \mathcal{V}_2 \) can reach \( p_{i_2} \) and no node in \( \mathcal{V}_2' \) can reach \( p_{i_2} \). Then \( \mathcal{V}_2' \) is closed and nonempty.

Next, select any node in \( \mathcal{V}_2' \), say \( p_{i_3} = 4 \). Notice that \( p_{i_3} \) is globally reachable in the local subgraph with nodes \( \mathcal{V}_2' \). Thus we can partition \( \mathcal{V} \) as

\[
\mathcal{V} = \mathcal{W}_2 \cup \mathcal{W}_2' \cup \mathcal{V}_2' = \{2, 5, 8\} \cup \{1, 3, 7\} \cup \{4, 6\},
\]

where every node in \( \mathcal{W}_2 \) can reach some node in \( \mathcal{V}_2' \) and no node in \( \mathcal{W}_2' \) can reach \( \mathcal{V}_2' \).

Then \( \mathcal{W}_2' \) and \( \mathcal{V}_2' \) are disjoint closed subsets. \( \square \)

The next tool we need is **Geršgorin’s Theorem**. Consider a \( 4 \times 4 \) matrix:

\[
A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34} \\
a_{41} & a_{42} & a_{43} & a_{44}
\end{bmatrix}.
\]

If it were diagonal, its eigenvalues would be \( a_{11}, a_{22} \) etc. If the off-diagonal elements are very small, by continuity the eigenvalues are close to \( a_{11}, a_{22} \) etc. Now construct the following disk centred at \( a_{11} \):
Construct three other similar disks centred at the other diagonal entries. Then Geršgorin’s Theorem is that the eigenvalues are contained within the union of these disks:

\[
\{\text{eigs of } A\} \subset \bigcup_{i=1,\ldots,4} \left\{ \lambda : |\lambda - a_{ii}| \leq \sum_{j \neq i} |a_{ij}| \right\}
\]

This extends to \(n \times n\) matrices.

Consider a digraph and associated Laplacian \(L\).

**Lemma 10** \(0\) is an eigenvalue of \(L\), and the vector of 1s is a corresponding eigenvector. All other eigenvalues have positive real part.

**Proof** The first statement is because the row sums of \(L\) are zero. The second part is immediate from Geršgorin. \(\square\)

**Example** The lemma is illustrated by this example:

\[
D = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}, \quad A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{bmatrix}, \quad L = D - A = \begin{bmatrix}
1 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 1 \\
\end{bmatrix}
\]

\(L1 = 0\)

The matrix \(L\) has an eigenvalue at 0. For the others, apply Geršgorin to \(L\). There are two distinct disks:

\[|s - 1| \leq 1, \quad |s - 2| \leq 2.\]

These are in the right half-plane except where they touch the origin.

**Lemma 11** If the digraph is strongly connected, 0 is a simple eigenvalue of \(L\).
Proof Assume the digraph is strongly connected. We have $L = D - A$. The degree matrix is invertible; the adjacency matrix $A$ is nonnegative and irreducible. We have

$$L = D - A = D(I - D^{-1}A).$$

Define

$$\tilde{A} = D^{-1}A, \quad \tilde{L} = D^{-1}L = I - \tilde{A}.$$ 

Notice that $\tilde{A} \geq 0$ and the two matrices $A, \tilde{A}$ have zeros in exactly the same locations. Thus $\tilde{A}$ is irreducible too.

Then by Frobenius’ theorem, $\rho(\tilde{A})$ is a simple eigenvalue of $\tilde{A}$. Since $\tilde{A}$ has constant row sums, namely 1, in fact $\rho(\tilde{A}) = \|\tilde{A}\|_\infty = 1$. It follows that 0 is a simple eigenvalue of $\tilde{L} = I - \tilde{A}$. Therefore, $\tilde{L}$ has rank $n - 1$, hence so does $L = D\tilde{L}$. Thus 0 is a simple eigenvalue of $L$. \qed

Now for the key result:

**Lemma 12** The digraph has a globally reachable node iff 0 is a simple eigenvalue of $L$.

**Proof** ($\Rightarrow$) Assume the digraph has a globally reachable node. Let $V_1$ be the set of all globally reachable nodes; $r = \text{card}(V_1)$.

**Case 1** $r = n$ The digraph is strongly connected. Therefore, by Lemma 11, 0 is a simple eigenvalue of $L$.

**Case 2** $1 < r < n$ Running example: $n = 4, r = 2$

![Graph](image)

Write

$$V = V_1 \cup V_2; \quad V_1 = \{1, 2\}, \quad V_2 = \{3, 4\}.$$

Then

$$(\forall p_1 \in V_1, p_2 \in V_2) \ p_2 \to p_1, \ p_1 \not\to p_2.$$

So $D, A, L$ have the form

$$D = \begin{bmatrix} D_1 & 0 & 0 \\ 0 & D_3 \end{bmatrix}, \quad A = \begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix},$$

$$L = \begin{bmatrix} L_1 & 0 \\ 0 & L_3 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix},$$

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & -1 & 0 & 2 \end{bmatrix}.$$
The graph $G(A_1)$ is strongly connected and $D$ is invertible. By Lemma 11, 0 is a simple eigenvalue of $L_1$. Define

$$
\tilde{A} := D^{-1}A = \begin{bmatrix}
\tilde{A}_1 & 0 \\
\tilde{A}_2 & \tilde{A}_3
\end{bmatrix},
\quad
\tilde{L} := D^{-1}L = I - \tilde{A} = \begin{bmatrix}
\tilde{L}_1 & 0 \\
\tilde{L}_2 & \tilde{L}_3
\end{bmatrix};
$$

$$
\tilde{A} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0.5 & 0.5 & 0 & 0
\end{bmatrix},
\quad
\tilde{L} = \begin{bmatrix}
1 & -1 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
-0.5 & -0.5 & 1 & -1
\end{bmatrix}.
$$

Thus, $\tilde{A}$ is row stochastic; therefore so is $\tilde{A}^k$. E.g.,

$$
\tilde{A}^2 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0.5 & 0.5 & 0 & 0 \\
0.5 & 0.5 & 0 & 0
\end{bmatrix}.
$$

Furthermore, $\tilde{A}^n$ has the form

$$
\tilde{A}^n = \begin{bmatrix}
\tilde{A}_1^n & 0 \\
X & \tilde{A}_3^n
\end{bmatrix}.
$$

Since every node in $V_2$ can reach $V_1$ after $n$ steps, each row of $X$ has a positive entry. E.g.,

$$
\tilde{A}^4 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0.5 & 0.5 & 0 & 0 \\
0.5 & 0.5 & 0 & 0
\end{bmatrix}.
$$

Thus

$$
\|\tilde{A}_3^n\|_\infty < 1 \implies \rho(\tilde{A}_3^n) < 1
\implies \rho(\tilde{A}_3) < 1
\implies \tilde{L}_3 = I - \tilde{A}_3 \text{ is invertible.}
\implies 0 \text{ is a simple eigenvalue of } L.
$$

**Case 3**  $r = 1$ Running example: $n = 4, r = 1$

![Graph](image_url)

Write

$$
V = V_1 \cup V_2; \quad V_1 = \{1\}, \quad V_2 = \{2, 3, 4\}.
$$
Then
\[(\forall p_1 \in \mathcal{V}_1, p_2 \in \mathcal{V}_2) \quad p_2 \rightarrow p_1, \quad p_1 \not\rightarrow p_2.\]

So $D, A, L$ have the form

\[
D = \begin{bmatrix} 0 & 0 \\ 0 & D_3 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 \\ A_2 & A_3 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 \\ L_2 & L_3 \end{bmatrix};
\]

\[
D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & -1 & -1 & 2 \end{bmatrix}.
\]

Since $D$ is singular, it’s convenient to define

\[D_1 = 1, \quad A_1 = 1\]

and rewrite $L$ as ($L$ doesn’t change)

\[L = \begin{bmatrix} 0 & 0 \\ L_2 & L_3 \end{bmatrix} = \begin{bmatrix} D_1 & 0 \\ 0 & D_3 \end{bmatrix} - \begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix}.
\]

The rest of the proof proceeds as in Case 2.

$(\Leftarrow)$ Suppose the digraph does not have a globally reachable node. By Lemma 9, there exist two disjoint closed subsets of $\mathcal{V}$, say $\mathcal{V}_1, \mathcal{V}_2$.

Running example:

\[\begin{array}{c}
4 \\
\end{array} \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array} \begin{array}{c}
1 \\
2 \\
3 \\
\end{array}
\]

Write

\[\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3; \quad \mathcal{V}_1 = \{1\}, \mathcal{V}_2 = \{2, 3\}, \mathcal{V}_3 = \{4\}.
\]

Then $D, A, L$ have the form

\[
D = \begin{bmatrix} D_{11} & 0 & 0 \\ 0 & D_{22} & 0 \\ 0 & 0 & D_{33} \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \quad L = \begin{bmatrix} L_{11} & 0 & 0 \\ 0 & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix}.
\]

By Lemma 10, 0 is an eigenvalue of both $L_{11}$ and $L_{22}$. Hence, 0 is not a simple eigenvalue of $L$. \(\square\)
3.3.4 Proof of Theorem 1

(1. $\Rightarrow$) Assume $G$ does not have a globally reachable node. Let $F_i$ be any controllers. By Lemma 9, there are two disjoint closed sets of nodes.

Running example:

Write

$V = V_1 \cup V_2 \cup V_3; \ V_1 = \{1\}, V_2 = \{2, 3\}, V_3 = \{4\}.$

Start the robots like this:

$z_1(0) = a_1, \ z_2(0) = z_3(0) = a_2, \ z_4(0) = a_4,$

with $a_1 \neq a_2$. Then $z_1(t) = a_1 \ \forall t$ and $z_2(t) = z_3(t) = a_2 \ \forall t$. ($z_1$ is a beacon.) Hence no convergence to a common point—problem not solvable.

(1. $\Leftarrow \& 2.$) Assume $G$ has a globally reachable node. Apply $F_i = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}$ (row vector of 1s). Then $\dot{z} = Mz$, $M = -L$. By Lemma 12, 0 is a simple eigenvalue of $M$; by Lemma 10 all other eigenvalues of $M$ have $\text{Re} \lambda < 0$, and $1$ is a corresponding eigenvector. Thus $\exists z_{ss}$ such that $z(t) \rightarrow z_{ss}1$. \hfill \Box

3.3.5 Exercises

1. Choose a relatively complicated digraph that doesn’t have a globally reachable node. (Hint: Arrange that $L$ doesn’t have a simple zero eigenvalue.) Then use the proof of Lemma 9 to construct two disjoint closed sets of nodes.

2. If robot $i$ pursues the average of the neighbour robots’ directions, the system is

$\dot{z}_i(t) = (m_i)^{-1}\sum_{j \in N_i} [z_j(t) - z_i(t)], \ i = 1, \ldots, n.$

Try a few simulations. Prove convergence to a common point.

3.3.6 References

3.4 The Rendezvous Problem: Time-Varying Visibility Graph with Symmetry

In this section we assume the robots have identical onboard sensors, so if robot \( i \) can sense robot \( j \), then \( j \) can sense \( i \); symmetry. The difference in this section is that the visibility graph is time-varying.

3.4.1 Problem Statement

So consider again \( n > 1 \) robots:

\[
\dot{z}_i = u_i, \quad i = 1, \ldots, n
\]

Robot \( i \) senses the subset \( \mathcal{N}_i(t) \), with the symmetry condition

\[
j \in \mathcal{N}_i \iff i \in \mathcal{N}_j.
\]

The sensed variables are \( y_i \), a vector with components \( z_j - z_i, j \in \mathcal{N}_i(t) \). We will consider only one control law, namely,

\[
u_i(t) = \sum_{j \in \mathcal{N}_i(t)} [z_j(t) - z_i(t)], \quad i = 1, \ldots, n.
\] (3.2)

Example visibility graph \( n = 4 \)

\[
\begin{array}{ccc}
1 & 2 & 3 \\
\downarrow & \downarrow & \downarrow \\
4
\end{array}
\]

The adjacency matrix is

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0
\end{bmatrix},
\]

which is symmetric. We could, if we liked, drop the arrowheads and regard the graph is being \textbf{undirected}, but we choose not to do this since our Perron-Frobenius theory concerns digraphs. Instead, we regard the graph as being directed and \textbf{symmetric}.

Let \( \{\mathcal{G}_p : p \in \mathcal{P}\} \) denote the class of all possible symmetric digraphs defined on \( n \) vertices. The set \( \mathcal{P} \) is any convenient index set. E.g., for \( n = 2 \), \( \mathcal{P} \) has \( 2^1 = 2 \) elements and we can take \( \mathcal{P} = \{1, 2\} \):

\[
\begin{array}{ccc}
\mathcal{G}_1 & & \mathcal{G}_2 \\
\bullet & \bullet & \bullet \rightarrow \bullet
\end{array}
\]
For \( n = 3 \), there are 3 possible communication links, so \( \mathcal{P} \) has \( 2^3 = 8 \) elements. For \( n = 4 \), there are 6 possible communication links, so \( \mathcal{P} \) has \( 2^6 \) elements.

For a symmetric digraph, the concept of being strongly connected is a little simpler than in general, because all paths go in both directions. Recall a result from our section on time-invariant visibility graphs:

**Lemma 13** If a symmetric digraph \( \mathcal{G}_p \) is strongly connected, then 0 is a simple eigenvalue of its Laplacian \( L_p \) and the other eigenvalues have positive real parts.

At time \( t \), let the sensor digraph be \( \mathcal{G}_{p(t)} \), let \( D_{p(t)} \) be its valence matrix, let \( A_{p(t)} \) be its adjacency matrix, let \( L_{p(t)} \) be its Laplacian, and let \( M_{p(t)} = -L_{p(t)} \). Then (3.2) is equivalent to

\[
\dot{z} = M_{p(t)} z.
\] (3.3)

We choose to regard \( p(t) \) as a time-varying parameter. So we have a linear time-varying system. Note that \( D_p, A_p, L_p, M_p \) are all symmetric.

The signal \( p(t) \) switches among a finite number of values as \( t \) progresses: \( p : \mathbb{R} \rightarrow \mathcal{P} \)

\[
\begin{array}{c}
1 \quad 2 \\
\downarrow \longrightarrow \quad \downarrow \\
4 \quad 3 \\
\end{array}
\begin{array}{c}
1 \quad 2 \\
\downarrow \longrightarrow \quad \downarrow \\
4 \quad 3 \\
\end{array}
\]

It is assumed that chattering doesn’t occur, that is, that \( p(t) \) switches a finite number of times in every finite time interval. Can this assumption be justified—I don’t know.

### 3.4.2 Solution

Even though the sensor digraph is time-varying, we still have convergence to a common point as long as no robot ever gets dropped from the communication network.

**Theorem 16** Concerning (3.3), assume \( \mathcal{G}_{p(t)} \) is strongly connected for every \( t \geq 0 \). Then the centroid of the points \( z_1(t), \ldots, z_n(t) \) is stationary and every \( z_i(t), i = 1, 2, \ldots, n \) converges to this centroid.

The proof requires a preliminary result.
Lemma 14 Let $M \in \mathbb{R}^{n \times n}$ be a real symmetric matrix with eigenvalues satisfying $\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_2 < \lambda_1 = 0$. Let $\mathcal{E}_0$ denote the eigenspace for $\lambda_1 = 0$ and let $\mathcal{E}_1$ denote the orthogonal complement of $\mathcal{E}_0$. Then for every $x \in \mathcal{E}_1$,

$$x^T M x \leq \lambda_2 x^T x.$$ 

Proof A real symmetric matrix has only real eigenvalues and it has an orthonormal set of eigenvectors. Let $v_1, v_2, \ldots, v_n$ be an orthonormal set of eigenvectors of $M$ corresponding to the eigenvalues $\lambda_1 = 0, \lambda_2, \ldots, \lambda_n$. Then

$$M v_i = \lambda_i v_i,$$

so

$$M \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}.$$ 

This implies that

$$M = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}^T,$$

and hence that

$$M = \lambda_1 v_1 v_1^T + \cdots + \lambda_n v_n v_n^T.$$ 

Since $\lambda_1 = 0$,

$$M = \lambda_2 v_2 v_2^T + \cdots + \lambda_n v_n v_n^T.$$ 

So for $x \in \mathcal{E}_1$,

$$x^T M x = \lambda_2 x^T v_2 v_2^T x + \cdots + \lambda_n x^T v_n v_n^T x \leq \lambda_2 (x^T v_2 v_2^T x + \cdots + x^T v_n v_n^T x) = \lambda_2 x^T (v_2 v_2^T + \cdots + v_n v_n^T) x = \lambda_2 x^T x.$$ 

Proof of Theorem 16 From Lemma 13, for every $t$, 0 is a simple eigenvalue of $M_p(t)$ and all the others have negative real part. Furthermore, the vector $1$ of 1s is a common eigenvector. Thus $\mathcal{E}_0 = \text{span}\{1\}$ is a common eigenspace and its orthogonal complement, $\mathcal{E}_1$, is the sum of all the other eigenspaces, for all $t$. The trajectory looks like

$$z(t) = a 1 + w(t), \quad w(t) \in \mathcal{E}_1$$ 

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and it remains to show that \( w(t) \to 0 \). Since \( w(t) = z(t) - a1 \) and \( M_{p(t)} \mathbf{1} = 0 \),

\[
\dot{w}(t) = \dot{z}(t) = M_{p(t)} z(t) = M_{p(t)} w(t).
\]

We know that for every \( w(0) \in \mathcal{E}_1 \), the solution \( w(t) \in \mathcal{E}_1 \), \( \forall t \geq 0 \) i.e., \( \mathcal{E}_1 \) is a positively invariant set for the system \( \dot{w}(t) = M_{p(t)} w(t) \).

Choose the Lyapunov function

\[
V(w) = \frac{1}{2} w^T w.
\]

Take the derivative of \( V(w(t)) \) along the solution of \( \dot{w}(t) = M_{p(t)} w(t) \):

\[
\dot{V}(w(t)) = w^T(t) M_{p(t)} w(t).
\]

From Lemma 14,

\[
\dot{V}(w(t)) = w^T(t) M_{p(t)} w(t) \leq -W(w(t)),
\]

where \( W(w) := -(\max_p \lambda_{p2}) w^T w \), \( \lambda_{p2} \) is the largest nonzero eigenvalue of \( A_p \), and the max is over all \( p \) for which the visibility graph is connected. Thus

\[
W(w) > 0, \quad \forall w \in \mathcal{E}_1 - \{0\}, \quad \text{and} \quad W(0) = 0.
\]

Therefore, by the Lyapunov stability theorem for non-autonomous systems (e.g., Khalil), every trajectory starting in \( \mathcal{E}_1 \) converges to 0. \( \square \)

### 3.4.3 Conclusion

The key point in this section is that if the robots have identical sensors, then \( \mathcal{G}_p \) is symmetric, hence \( M_{p} \) is symmetric and there is a common Lyapunov function.

However, the result is not really applicable. In a sensible problem the graph is state-dependent, not time-dependent. We’ll return to this point again.

### 3.4.4 Exercises

1. Consider the time-varying system \( \dot{z} = M(t) z \). Suppose \( M(t) \) can take on only two values, \( M_1, M_2 \). Find or look up an example where the eigenvalues of \( M_1, M_2 \) all have negative real parts, yet the origin of \( \dot{z} = M(t) z \) is not stable.

### 3.4.5 References

3.5 Using Convexity

This section reviews the work of Moreau. This work simplifies and generalizes Jadbabaie, Lin, Morse. More importantly, it introduces the relevance of convexity.

3.5.1 Problem Setup

**Running example** The setup is discrete time: $t \in \mathbb{N}$. Suppose at time $t$ the visibility graph $\mathcal{G}(t)$ is

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}
\]

Then the (out-)degree matrix and adjacency matrix are

\[
D(t) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad A(t) = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

Suppose each robot moves to the centroid location of itself and the ones it can sense:

\[
\begin{align*}
z_1(t+1) &= \frac{1}{2}[z_1(t) + z_2(t)] \\
z_2(t+1) &= \frac{1}{3}[z_1(t) + z_2(t) + z_3(t)] \\
z_3(t+1) &= z_3(t) \\
z_4(t+1) &= z_4(t).
\end{align*}
\]

This kind of strategy makes sense, for example, when $z_i$ is actually a real variable, such as a heading angle. Rewrite the equations as

\[
\begin{align*}
z_1(t+1) + z_1(t+1) &= z_1(t) + z_2(t) \\
z_2(t+1) + 2z_2(t+1) &= z_2(t) + z_1(t) + z_3(t) \\
z_3(t+1) + 0z_3(t+1) &= z_3(t) \\
z_4(t+1) + 0z_4(t+1) &= z_4(t).
\end{align*}
\]

In vector form

\[
z(t+1) + D(t)z(t+1) = z(t) + A(t)z(t),
\]

that is,

\[
z(t+1) = [I + D(t)]^{-1} [I + A(t)] z(t).
\]

In this example,

\[
[I + D(t)]^{-1} [I + A(t)] = \begin{bmatrix}
1/2 & 1/2 & 0 & 0 \\
1/3 & 1/3 & 1/3 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

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The eigenvalues of this matrix are $0, \frac{1}{2} + \frac{1}{3}, 1, 1$. Two are at $\lambda = 1$. Meaning?

It’s worth noting that $z$, $x$, and $y$ all satisfy the same equation:

\[
\begin{align*}
z(t + 1) &= M(t)z(t) \\
x(t + 1) &= M(t)x(t) \\
y(t + 1) &= M(t)y(t) \\
M(t) &= [I + D(t)]^{-1}[I + A(t)].
\end{align*}
\]

As time goes by, the visibility graph $G(t)$ may change and consequently $D(t), A(t)$ are time-varying. The model is therefore linear time-varying:

\[
z(t + 1) = M(t)z(t), \quad M(t) = [I + D(t)]^{-1}[I + A(t)].
\]

### 3.5.2 The Theorem

**Theorem 17** Suppose $G(t)$ has the following property: There is a positive integer $T$ such that, for every $t_0$, $\bigcup_{t=t_0}^{t_0+T} G(t)$ has a globally reachable node. Then the robots converge to a common point.

Thus, there need not always be a globally reachable node—that is, connectivity may be occasionally lost—as long as persistently some node becomes globally reachable for the union visibility graph over a period. Note that $T$ is independent of $t_0$.

Moreau’s proof of Theorem 17 is quite complicated and we won’t do it in detail. Instead, we’ll look at the main idea, which involves the evolution of a convex set.

The simplest example is 2 robots. Suppose $G(t)$ versus time looks like this:

1
2
3
4
5
6
\quad \text{etc.}

The union of the first three graphs is

For this visibility graph pattern, the time plot of both $x$ and $y$ components would look like this:
3.5.3 Proof Technique

Introduce the sensed neighbours, $N_i(t)$. For the running example we have

\begin{align*}
N_1(t) &= \{2\}, & N_2(t) &= \{1, 3\}, & N_3(t) &= \emptyset, & N_4(t) &= \emptyset.
\end{align*}

We now look at some properties of the controlled system.
A robot won’t move if it’s collocated with all its sensed neighbours; running example:

\[ M(t) = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad \begin{bmatrix}
\ast \\
a \\
\ast \\
a
\end{bmatrix} = M(t) \begin{bmatrix}
a \\
a \\
a \\
a
\end{bmatrix} = \begin{bmatrix}
a \\
a
\end{bmatrix}.
\]

**Property 1: Stationarity** If $\{z_i(t), N_i(t)\}$ are collocated, then $z_i(t + 1) = z_i(t)$.

For the next property we need a bit of convexity theory. Let $p_1, p_2$ be two points in the plane. The set

\[ \{\lambda_1 p_1 + \lambda_2 p_2 : \lambda_i \geq 0, \lambda_1 + \lambda_2 = 1\} \]

is the line segment

\[ \begin{array}{c}
p_1 \\
\bullet
\end{array} \quad \begin{array}{c}
p_2 \\
\bullet
\end{array} \]

Let $p_1, p_2, p_3$ be three points in the plane. The set

\[ \{\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 : \lambda_i \geq 0, \lambda_1 + \lambda_2 + \lambda_3 = 1\} \]

is the triangle (or line or point)
The **convex hull** of a subset $S$ of $\mathbb{C}$, or $\mathbb{R}^2$, is the intersection of all convex sets containing $S$. If $S$ has only finitely many points, $p_1, \ldots, p_N$, then the convex hull is

$$
\text{conv}(S) = \left\{ \sum_{i=1}^{N} \lambda_i p_i : \lambda_i \geq 0, \sum_{i=1}^{N} \lambda_i = 1 \right\}.
$$

Let’s turn to the running example:

$$
M(t) = \begin{bmatrix}
1/2 & 1/2 & 0 & 0 \\
1/3 & 1/3 & 1/3 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

The point $z_2(t+1)$ lies at the centroid of $z_1(t), z_2(t), z_3(t)$. Thus, if $z_1(t), z_2(t), z_3(t)$ are not collinear, then they form a triangle and $z_2(t + 1)$ lies in the interior of it. If the points $z_1(t), z_2(t), z_3(t)$ are collinear but not collocated, then they form a line segment and $z_2(t + 1)$ lies in the interior of it.

In general again, consider a convex set $S$ in $\mathbb{C}^n$. It lies in an affine space of least dimension. (An **affine space** is the translate of a subspace.) That affine space is the **affine hull** of $S$. That affine hull has a topology. The **relative interior** of $S$ is the interior with respect to the affine hull, i.e., the union of all subsets of $S$ that are open in the affine hull.

**Property 2: Strict convexity** If \{z_i(t), N_i(t)\} are not collocated, then $z_i(t+1)$ lies in the relative interior of $\text{conv} \{z_i(t), N_i(t)\}$.

For a vector $p \in \mathbb{C}^n$, let $V(p)$ denote $\text{conv}\{p_1, \ldots, p_n\}$, the convex hull of the components of $p$. Thus $V$ is a function that maps the state space $\mathbb{C}^n$ into a subset of $\mathbb{C}$. It’s a set-valued mapping. Property 2 implies that $V$ is non-increasing along solutions of the state equation

$$
z(t + 1) = M(t)z(t).
$$

Running example:
E.g.,
\[ z_1(t+1) \in \text{conv}\{z_1(t), z_2(t)\} = \text{conv}\{z_1(t), N_1(t)\} \subset V(z(t)) \]
Likewise for \(z_2, z_3, z_4\), and so \(V(z(t+1)) \subset V(z(t))\). The general statement is
\[ V(z(t+1)) \subset V(z(t)), \quad \forall t \in \mathbb{N} \]
i.e.,
\[ V(M(t)z) \subset V(z), \quad \forall t \in \mathbb{N}, z \in \mathbb{C}^n. \]
The function \(V\) acts as a sort of Lyapunov function.

Now the inclusion \(V(z(t+1)) \subset V(z(t))\) isn’t enough for convergence of the robots to a common point, because conceivably we could get \(V(z(t+1)) = V(z(t))\) at some time without \(V(z(t))\) being a singleton. So now we need to use the assumption that there is a positive integer \(T\) such that, for every \(t_0\), \(\bigcup_{t=t_0}^{t_0+T} \mathcal{G}(t)\) has a globally reachable node.

**Some Proof Details, \(n = 2\)**

Consider the example

We need \(T \geq 2\) for \(\mathcal{G}(t)\) as shown. Let \(t_0 = 1\). If \(z_1(t_0) = z_2(t_0)\), we’re done. So assume \(z_1(t_0) \neq z_2(t_0)\). Over the interval \([t_0, t_0 + T + 1]\) at least one robot moves. (The +1 is needed because reachability may occur at \(t_0 + T\).) Thus \(V(t_0 + T + 1)\) is a proper subset of \(V(t_0)\). We could show \(\exists \beta > 0\) such that
\[ \text{length}\{V(t_0)\} - \text{length}\{V(t_0 + T + 1)\} \leq \beta. \]
That is, the length of \(V(t)\) decreases to 0 in the limit.
Some Proof Details, \( n = 3 \)

Suppose the graph has the stated property. Suppose \( z_1(t_0), z_2(t_0), z_3(t_0) \) are not collinear. Over the interval \([t_0, t_0 + T + 1]\), at least two robots move—toward the one that’s globally reachable. Again we could show \( \exists \beta > 0 \) such that

\[
\text{diameter}\{V(t_0)\} - \text{diameter}\{V(t_0 + T + 1)\} \leq \beta.
\]

One interesting twist is shown here:

Suppose \( T = 1 \) and \( \mathcal{G}(t_0), \mathcal{G}(t_0 + 1) \) are as shown on the left. Robot 1 is globally reachable for \( \mathcal{G}(t_0) \cup \mathcal{G}(t_0 + 1) \). Suppose \( z_2, z_3 \) are collocated at \( t_0 \), as shown on the right. Then \( V(t_0) \) and \( V(t_0 + 2) = V(t_0 + T + 1) \) are equal—there’s no shrinking over that period. We have to wait longer. How much longer is left for you to think about.

3.5.4 Counterexample

Moreau’s paper has other interesting results, such as the following example. Consider 3 robots with four possible visibility graphs:

The corresponding dynamic matrices are

\[
M_a = \begin{bmatrix}
1 & 0 & 0 \\
1/2 & 1/2 & 0 \\
0 & 0 & 1
\end{bmatrix}, \text{ etc.}
\]

Notice that, for example, the union of \( a \) and \( d \) has a globally reachable node—1.
We’re going to construct the time-function \( G(t) \) by means of an infinite string in the letters \( a, b, c, d \). For example, the string \( a^2b^3d \cdots \) would correspond to \( G(t) \) as follows:

\[
\begin{array}{c|cccccccc}
  t & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\
  G(t) & a & a & b & b & b & d & c & \cdots \\
\end{array}
\]

First define the finite strings

\[ B_k = a^{2k}bc^{2k+1}d, \quad k = 0, 1, 2, \ldots \]

Then define the infinite string

\[
B = B_0B_1B_2\ldots = \underbrace{bcd}_{t_1} \underbrace{a^2bc^3d}_{t_2} \underbrace{a^4bc^5d}_{t_3} \ldots
\]

Finally, let \( G(t) \) correspond to \( B \).

Notice that this is true:

For every \( t_0 \) there is a positive integer \( T \) such that \( \bigcup_{t=t_0}^{t_0+T} G(t) \) has a globally reachable node.

But this is not:

There is a positive integer \( T \) such that, for every \( t_0 \), \( \bigcup_{t=t_0}^{t_0+T} G(t) \) has a globally reachable node.

Let \( z_1(1) = 0, z_2(1) = z_3(1) = 1 \). Then the robots do not converge to a common point.

**Proof** Define the time instants

\[
\begin{align*}
t_1 &= 2 \\
t_{k+1} &= t_k + k + 1.
\end{align*}
\]

This sequence of times is linked to the string \( B \), and hence \( G(t) \), as follows:

\[
B = B_0B_1B_2\ldots = \underbrace{bcd}_{t_1} \underbrace{a^2bc^3d}_{t_2} \underbrace{a^4bc^5d}_{t_3} \ldots
\]

\[
\begin{array}{cccccccccc}
b & c & d & a & a & b & c & c & d & a & a & \cdots \\
\hline
  t_1 & t_2 & t_3 & t_4 & \cdots
\end{array}
\]

Let us look at the motion ...
We have
\[ v(1) = \frac{1}{2} \]
\[ v(2) = v(1) - \frac{1}{2^2} v(1) \]
\[ v(3) = v(2) - \frac{1}{2^3} v(2) \]
and in general
\[ v(k + 1) = v(k) - \frac{1}{2^{k+1}} v(k). \]

Then
\[
\lim_{k \to \infty} v(k) = v(1) - \sum_{k=1}^{\infty} [v(k) - v(k + 1)] \\
= \frac{1}{2} - \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} v(k) \\
> \frac{1}{2} - \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} \quad \text{since } v(k) < 1 \\
= \frac{1}{2} - \frac{1}{2} \left( \frac{1}{2} + \frac{1}{4} + \cdots \right) \\
= 0.
\]

So there's a permanent gap between robots 1 and 3.
3.5.5 Conclusions

On the positive side, Moreau introduced convexity into the problem—an important conceptual contribution. (Jadbabaie et al. had used a theorem of Wolfowitz.) But Moreau’s complete proof is very hard to follow. Furthermore, the graph condition is not constructive: When does the graphical condition hold? We would prefer a condition on $G(t_0)$ only. Finally, the extension to continuous time is very difficult technically—continuous-time evolution of the convex hull of the robots.

3.5.6 Exercises

1. Consider a directed graph with degree matrix $D$ and adjacency matrix $A$. Prove that

\[(I + D)^{-1}(I + A)\mathbf{1} = \mathbf{1}.

Thus 1 is an eigenvalue and 1 a corresponding eigenvector. Prove that the other eigenvalues satisfy $|\lambda| < 1$.

3.5.7 References


2. L. Moreau, “Time-dependent unidirectional communication in multiagent systems,” June 30, 2003; accepted *IEEE TAC*.
3.6 The Circumcentre Control Law

In this section we make a giant step over the preceding ones: We take a more realistic sensor model and assume each robot can see only a given distance.

3.6.1 Introduction

In their seminal paper, Jadbabaie et al. study a system of “boids” moving at constant speed in the plane. Each boid has a heading angle, $\theta_i$, evolving in discrete time. Also, at time $t$ boid $i$ can see a set $\mathcal{N}_i(t)$ of neighbours, and this leads to a time-varying graph $G(t)$. Jadbabaie et al. assume if boid $i$ can see boid $j$, then $j$ can see $i$. Thus, $G(t)$ is undirected. The paper studies a single control strategy, namely, at time $t + 1$ boid $i$ changes its heading to the average heading at time $t$ of itself and its neighbours. The paper proves that all the heading angles converge to a common value (similar to rendezvous) provided $G(t)$ has a connectedness property over time, namely, there exists $T > 0$ such that the union graph $\bigcup_{t_0 \leq t \leq t_0 + T} G(t)$ is connected for all $t_0$. Here, the union of graphs with the same node set is obtained by taking the union of the edges. Unfortunately, the condition on $G(t)$ is not checkable—it would require an infinite time simulation. The proof uses a theorem of Wolfowitz on ergodicity (1963).

In the theorems in Jadbabaie et al. the vector $\theta(0)$ of initial heading angles is fixed. This allows the visibility graph to be a function of $t$ alone, and not both $t$ and $\theta(0)$. In actuality, for a sensible model of limited visibility, the graph is state dependent, not time dependent. To see this, let us return to our robots. Suppose robot $i$ has an omnidirectional camera of range $R_i$. The set of visible neighbours of robot $i$ is

$$\mathcal{N}_i(z) = \{ j : j \in \{1, \ldots, n\}, \ |z_j - z_i| \leq R_i \}.$$ 

Then there’s an edge in the visibility graph from $i$ to $j$ iff $j \in \mathcal{N}_i(z)$, and therefore the visibility graph is a function of $z$, $G(z)$. If the control strategy is given and if the state $z(t)$ evolves uniquely from $z(0)$, the visibility graph is a function of time, $G(z(t))$.

Now we turn to the **circumcentre control law**.

**Example** Consider six robots, with omnidirectional cameras of identical ranges, positioned at $t = 0$:

![Diagram showing visibility fields for robots 1 and 2. The fields are shown as overlapping circles, indicating the range of each camera.]
Thus the neighbour sets at $t = 0$ are

$$
N_1 = \{2\}, \quad N_2 = \{1, 4\}, \quad N_3 = \{4\}, \quad N_4 = \{2, 3, 5\}, \quad N_5 = \{4, 6\}, \quad N_6 = \{5\}.
$$

The circumcentre control law is defined as follows: Robot 1 has one neighbour, robot 2. Let $Z_1 = \{z_1, z_2\}$ and let $c_1$ denote the circumcentre of $Z_1$—the centre of the smallest circle containing $Z_1$. Then set $u_1 = c_1 - z_1$:

$$
\dot{z}_1 = c_1 - z_1.
$$

(In the picture, the little arrow is $u_1$ translated from the origin to $z_1$.) So robot 1 moves towards the centre at $t = 0$: $\dot{z}_1 = c_1 - z_1$. Actually, in this case where 1 sees only 2, clearly $c_1 = (z_1 + z_2)/2$, so at $t = 0$

$$
\dot{z}_1 = \frac{1}{2}(z_2 - z_1).
$$

Similarly, let $c_2$ denote the circumcentre of the set $\{z_2\} \cup \{z_j : j \in N_2\}$ and define $u_2 = c_2 - z_2$:

$$
\dot{z}_2 = c_2 - z_2.
$$

And so on.

These control laws can actually be implemented using onboard cameras, that is, relative positions, by translation. For example, for robot 2, the relative positions $\{z_1 - z_2, z_4 - z_2\}$ are sensed.
Let $Z'_2$ denote the set of points $\{0, z_1 - z_2, z_4 - z_2\}$ (the translate of $\{z_2, z_1, z_4\}$ by $-z_2$), and let $c'_2$ denote the circumcentre of $Z'_2$. Then define $u_2 = c'_2$.

Let’s look at $u_1$ again. The set $Z_1$ equals $\{z_1, z_2\}$ and so the circumcentre $c_1$ of $Z_1$ is a function of $z$; write $c_1(z)$. It turns out that $c_1(z)$ is continuous in $z$, but not Lipschitz continuous—see below. In this way, the robots’ motions are governed by the coupled equations

$$\begin{align*}
\dot{z}_1 &= u_1(z) = c_1(z) - z_1 \\
&\vdots \\
\dot{z}_6 &= u_6(z) = c_6(z) - z_6,
\end{align*}$$

or in aggregate form $\dot{z} = u(z)$, where the vector field $u(z)$ is only continuous, not Lipschitz. Thus uniqueness of a solution is not guaranteed. In what follows, a statement about a solution should be interpreted as applying to all solutions if indeed there is more than one.

### 3.6.2 General Results

The fact that the circumcentre control law isn’t a Lipschitz continuous function causes difficulty in its use, as we’ll see later. So it’s perhaps of interest to see a proof. Reference:


**Lemma 15** The circumcentre control law isn’t Lipschitz continuous.

**Proof** Construct three points $\{p_1, p_2, p_3\}$ and their circumcentre $c$, and three perturbed points $\{p'_1, p'_2, p'_3\}$ and their circumcentre $c'$, like this:

![Diagram](image)

Define the vectors

$$p = (p_1, p_2, p_3), \quad p' = (p'_1, p'_2, p'_3).$$
We’ll show that the ratio
\[ \frac{|c - c'|}{\|p - p'|} \]
is not bounded by a constant. This proves the mapping \( p \mapsto c \) isn’t Lipschitz.

Let the radii of the circles be 1 and define \( x = |c - c'|, \ y = |p_2 - p'_2| \). Since \( p_1 \) didn’t move and \( |p_2 - p'_2| = |p_3 - p'_3| \),
\[ \|p - p'\| = \sqrt{2}y. \]

Now look at

Define the angle \( \theta \). Then we have the lengths \( p_1q = \cos \theta, \ q\overline{c'} = \sin \theta \). Thus \( \overline{qc} = 1 - \cos \theta \), so by Pythagoras on the small triangle \( qcc' \)
\[ x^2 = (1 - \cos \theta)^2 + \sin^2 \theta = 2(1 - \cos \theta), \]
and therefore \( \overline{qc} = x^2/2 \).

By Pythagoras again on the triangle \( qcc' \), the length of \( \overline{qc} \) equals \( x\sqrt{1 - \frac{x^2}{4}} \). Finally, apply Pythagoras to triangle \( qp_2c' \):
\[ (y + 1)^2 = \left(1 + \frac{x^2}{2}\right)^2 + x^2 \left(1 - \frac{x^2}{4}\right) = 2x^2 + 1. \]
Thus we have
\[ x = \sqrt{\frac{1}{2}y^2 + y}, \]
and so
\[ \frac{|c - c'|}{\|p - p'|} = \frac{x}{\sqrt{2}y} = \frac{1}{2} \sqrt{1 + \frac{2}{y}}. \]
However, the great thing about the circumcentre law is that it preserves connectivity of the visibility graph. Thus we will only have to assume that the visibility graph is connected at $t = 0$.

In fact, under the circumcentre control law, no links are dropped (though the distances between some neighbours may increase), so if $G(0)$ is connected, then $G(t)$ is connected for all $t > 0$.

Of course, new links may form: As the robots rendezvous, eventually the graph becomes complete.

**Lemma 16** Under the circumcentre control law, over time no links are dropped in the visibility graph.

**Proof** For this proof it’s more convenient to view a robot position $z_i$ as a vector in $\mathbb{R}^2$ instead of a complex number. Let $t \geq 0$ be arbitrary. Let $V_{ij}(z(t))$ denote the distance-squared between two neighbour robots $i$ and $j$, and let $V(z(t))$ denote the maximum distance-squared between any two neighbours:

$$V(z(t)) = \max_i \max_{j \in N_i(z(t))} V_{ij}(z(t)).$$

Let $\mathcal{I}(z(t))$ denote the set of pairs of indices where the maximum is attained; that is, $(i, j) \in \mathcal{I}(z(t))$ iff robots $i$ and $j$ are neighbours of maximum distance apart among all neighbours. Thus

$$V(z(t)) = \max_{(i, j) \in \mathcal{I}(z(t))} V_{ij}(z(t)).$$

We would like to show that $d/dt V(z(t)) \leq 0$. Unfortunately, $V(z(t))$ is not differentiable. We need some non-smooth analysis—the upper Dini derivative:

$$D^+ V(z(t)) = \limsup_{\tau \to 0^+} \frac{V(z(t + \tau)) - V(z(t))}{\tau}.$$

Then, it is a fact that

$$D^+ V(z(t)) = \max_{(i, j) \in \mathcal{I}(z(t))} \frac{d}{dt} V_{ij}(z(t)).$$

(3.4)

In this way we get

$$D^+ V(z(t)) = \max_{(i, j) \in \mathcal{I}(z(t))} \frac{d}{dt} \|z_i(t) - z_j(t)\|^2$$

$$= \max_{(i, j) \in \mathcal{I}(z(t))} 2\langle z_i(t) - z_j(t), \dot{z}_i(t) - \dot{z}_j(t)\rangle$$

$$= \max_{(i, j) \in \mathcal{I}(z(t))} 2\langle z_i(t) - z_j(t), u_i(t) - u_j(t)\rangle$$

$$= \max_{(i, j) \in \mathcal{I}(z(t))} 2\{\langle z_i(t) - z_j(t), u_i(t)\rangle + 2\langle z_j(t) - z_i(t), u_j(t)\rangle\}$$

$$\leq \max_{(i, j) \in \mathcal{I}(z(t))} 2\langle z_i(t) - z_j(t), u_i(t)\rangle$$

$$+ \max_{(i, j) \in \mathcal{I}(z(t))} 2\langle z_j(t) - z_i(t), u_j(t)\rangle.$$

To conclude that

$$D^+ V(z(t)) \leq 0,$$

(3.5)

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we’ll show that
\[
\max_{(i,j) \in I(z(t))} \langle z_i(t) - z_j(t), u_i(t) \rangle \leq 0
\]  
(3.6)
and
\[
\max_{(i,j) \in I(z(t))} \langle z_j(t) - z_i(t), u_j(t) \rangle \leq 0.
\]  
(3.7)

To illustrate the argument, suppose \((1,j) \in I(z(t))\) for some \(j\), that is, the maximum separation between robot neighbours occurs for robot 1 (and perhaps others). Suppose the neighbours of robot 1 are robots 2, 3, 4, 5.

\[z_1\] \[z_2\] \[z_3\] \[z_4\] \[z_5\]

\[u_1\]

See the circumcentre control vector \(u_1\) (translated to \(z_1\)). The figure shows three neighbour robots—2, 3, and 4—on the smallest encompassing circle. Now in the figure

\[z_1\] \[z_2\] \[z_3\] \[z_4\] \[z_5\]

\[u_1\]

construct the line as shown through \(z_1\) perpendicular to \(u_1\), and using this line as diameter, draw a second circle. In the shaded crescent there must be a neighbour of robot 1, for otherwise the encompassing circle in the figure would be smaller (in fact it would be the unshaded circle). Consider the robot in the shaded crescent that is maximum distance from robot 1; in the figure it is robot 3. The angle between the vectors \(u_1\) and \(z_1 - z_3\) is greater than \(\pi/2\). Therefore
\[
\langle z_1(t) - z_3(t), u_1(t) \rangle \leq 0
\]
and so
\[
\max_{(1,j) \in I(z(t))} \langle z_1(t) - z_j(t), u_1(t) \rangle \leq 0.
\]

This proves (3.6), and (3.7) follows from this.
Finally, from (3.4) and (3.5), if two neighbours \( i \) and \( j \) are maximum distance apart (among all neighbours), then \( \frac{d}{dt}V_{ij}(z(t)) \leq 0 \) and so the distance between them is non-increasing. \( \square \)

Here’s the main result that the circumcentre control law solves the rendezvous problem:

**Theorem 18**  Suppose \( z(0) \) is such that \( G(z(0)) \) is connected. Under the circumcentre control law, the robots rendezvous.

The proof uses LaSalle’s theorem. Here we want to discuss the ideas without the details.

**Ideas for a Proof**  We’re given that \( G(z(0)) \) is connected. By Lemma 16, \( G(z(t)) \) is connected for all \( t > 0 \). Now \( G(z(t)) \) is either fixed or it’s not. Suppose not. Then at some time a new link appears (no link is dropped). After this, \( G(z(t)) \) is either fixed or it’s not. Suppose not. Then at some time, another new link appears. Since there are only finitely many nodes, this process must stop. Thus we may assume without loss of generality that \( G(z(t)) \) is fixed and connected for all \( t \geq 0 \). (We don’t assume the graph is complete, but it must actually be so, since the robots rendezvous.)

Bring in the example

![Diagram of a constellation with labeled nodes 1 through 6 showing the circumcentre control law in action.](image)

for illustrative purposes. The picture

shows the constellation at \( t = 0 \), its convex hull \( \text{co}\{z(0)\} \), and the instantaneous velocities \( u_i(z(0)) \) of the robots at the vertices. Even though the vector fields \( u_i(z(0)) \) point into \( \text{co}\{z(0)\} \), we can’t conclude that \( z_i(t) \in \text{co}\{z(0)\} \) because we don’t have Lipschitz continuity. So it’s problematic to prove even that a solution \( z(t) \) is bounded.

Let \( a \) be an arbitrary point in the plane and define the function \( V^a(z) \) to be the distance squared from \( a \) to the farthest \( z_i \). (Again, we take the plane \( \mathbb{R}^2 \).)
Assume for simplicity that the farthest-away robot doesn’t change, that it’s always robot 3. Then $V^a(z)$ is differentiable and
\[
\frac{d}{dt} V^a(z(t)) = \frac{d}{dt} \|z_3(t) - a\|^2 = 2\langle u_3(z(t)), z_3(t) - a \rangle.
\]
From the vector orientations in the figure, $\langle u_3(z(t)), z_3(t) - a \rangle \leq 0$. Thus $V^a(z(t))$ is nonincreasing, and this kind of argument shows that $z(t)$ is defined for all $t > 0$ and is bounded.

Now invoke LaSalle’s theorem. The solution converges to the largest invariant manifold $M$ in \{ $z : \dot{V}^a(z) = 0$ \}. To see what this manifold is, let $z(0) \in M$. Continuing with the assumption that the farthest-away $z_i(0)$ from $a$ is $z_3(0)$, we have that $u_3(z(0))$ and $z_3(0) - a$ are orthogonal. Looking at the figure we conclude that $u_3(z(0)) = 0$; for if $u_3(z(0)) \neq 0$ then $z_4(0)$, the only neighbour of $z_3(0)$, is farther from $a$ than is $z_3(0)$. Since $u_3(z(0)) = 0$, then $z_3$ and $z_4$ must be collocated at $t = 0$. If $z_2(0)$ and $z_5(0)$, the neighbours of $z_4(0)$, are not also collocated with $z_4(0)$, then $z_4(t)$ will move away from $z_3(0)$, which is impossible since $M$ is invariant. Using this kind of argument, one can prove that for $z \in M$, all $z_i$ are equal.

A rigorous proof is considerably more complicated since $V^a(z)$ is not actually differentiable. □

In 2003 Tanner et al. studied a group of autonomous mobile continuous-time boids. They proposed distributed control laws, involving potential functions, that achieve convergence to a common heading while avoiding collisions. Persistent connectivity of the visibility graph is assumed rather than guaranteed. Reference


### 3.6.3 Numerical Issues

Consider a set of distinct points, $p_1, \ldots, p_n$, in $\mathbb{C}$ and consider their circumcircle, $\mathcal{C}$. Either there are two points on $\mathcal{C}$ diametrically opposite, in which case the centre is easily calculated as the midpoint between them, or there are three points on $\mathcal{C}$ spanning an arc of more than $\pi$ radians; in this case the centre lies within the triangle formed from the three points. How to find these boundary points is not discussed here.

We turn to the problem of computing the centre, $c$, of the circle in the latter case. The setup is
where the three points lie on a circle and the centre of the circle lies inside the triangle. The centre is in the convex hull of \( p_0, p_1, p_2 \) and therefore can be parametrized as

\[
c = \lambda_0 p_0 + \lambda_1 p_1 + \lambda_2 p_2,
\]

where \( \lambda_i \geq 0 \) and

\[
\lambda_0 + \lambda_1 + \lambda_2 = 1.
\]

To see this, shift \( p_0 \) to the origin. Then \( c - p_0 \) is in the convex hull (line) of two points, \( \alpha_1(p_1 - p_0), \alpha_2(p_2 - p_0) \):

\[
c - p_0 = \lambda \alpha_1(p_1 - p_0) + (1 - \lambda) \alpha_2(p_2 - p_0),
\]

and hence

\[
c = [1 - \lambda \alpha_1 - (1 - \lambda) \alpha_2]p_0 + \lambda \alpha_1 p_1 + (1 - \lambda) \alpha_2 p_2,
\]

which has the form (3.8).

The centre satisfies the two equations

\[
|p_2 - c| = |p_0 - c|,
|p_1 - c| = |p_0 - c|.
\]

Thus

\[
(p_2 - c)(\bar{p}_2 - \bar{c}) = (p_0 - c)(\bar{p}_0 - \bar{c})
(p_1 - c)(\bar{p}_1 - \bar{c}) = (p_0 - c)(\bar{p}_0 - \bar{c})
\]
The solution of this equation gives $c$
or
This gives
\[|p_2|^2 - 2\text{Re}(c\bar{p}_2) + |c|^2 = |p_0|^2 - 2\text{Re}(c\bar{p}_0) + |c|^2\]
\[|p_1|^2 - 2\text{Re}(c\bar{p}_1) + |c|^2 = |p_0|^2 - 2\text{Re}(c\bar{p}_0) + |c|^2.\]

This gives
\[|p_2|^2 - 2\text{Re}(c\bar{p}_2) = |p_0|^2 - 2\text{Re}(c\bar{p}_0)\]
\[|p_1|^2 - 2\text{Re}(c\bar{p}_1) = |p_0|^2 - 2\text{Re}(c\bar{p}_0),\]
or
\[2\text{Re}[c(\bar{p}_0 - \bar{p}_2)] = |p_0|^2 - |p_2|^2\]
\[2\text{Re}[c(\bar{p}_0 - \bar{p}_1)] = |p_0|^2 - |p_1|^2,\]

Now bring in (3.8) and (3.9):
\[2\text{Re}[(\lambda_0 p_0 + \lambda_1 p_1 + \lambda_2 p_2)(\bar{p}_0 - \bar{p}_2)] = |p_0|^2 - |p_2|^2\]
\[2\text{Re}[(\lambda_0 p_0 + \lambda_1 p_1 + \lambda_2 p_2)(\bar{p}_0 - \bar{p}_1)] = |p_0|^2 - |p_1|^2\]
\[\lambda_0 + \lambda_1 + \lambda_2 = 1.\]

Thus
\[\lambda_0 2\text{Re}[p_0(\bar{p}_0 - \bar{p}_2)] + \lambda_1 2\text{Re}[p_1(\bar{p}_0 - \bar{p}_2)] + \lambda_2 2\text{Re}[p_2(\bar{p}_0 - \bar{p}_2)] = |p_0|^2 - |p_2|^2\]
\[\lambda_0 2\text{Re}[p_0(\bar{p}_0 - \bar{p}_1)] + \lambda_1 2\text{Re}[p_1(\bar{p}_0 - \bar{p}_1)] + \lambda_2 2\text{Re}[p_2(\bar{p}_0 - \bar{p}_1)] = |p_0|^2 - |p_1|^2\]
\[\lambda_0 + \lambda_1 + \lambda_2 = 1.\]

This can be written as
\[
\begin{bmatrix}
2\text{Re}[p_0(\bar{p}_0 - \bar{p}_2)] & 2\text{Re}[p_1(\bar{p}_0 - \bar{p}_2)] & 2\text{Re}[p_2(\bar{p}_0 - \bar{p}_2)] \\
2\text{Re}[p_0(\bar{p}_0 - \bar{p}_1)] & 2\text{Re}[p_1(\bar{p}_0 - \bar{p}_1)] & 2\text{Re}[p_2(\bar{p}_0 - \bar{p}_1)] \\
1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
\lambda_0 \\
\lambda_1 \\
\lambda_2
\end{bmatrix}
= 
\begin{bmatrix}
|p_0|^2 - |p_2|^2 \\
|p_0|^2 - |p_1|^2 \\
1
\end{bmatrix}.
\]

The solution of this equation gives $c$ via (3.8).

What about numerical sensitivity of this procedure? Consider the example

The points form an isosceles triangle: $p_0 = 0, p_1 = je^{-j\varepsilon}, p_2 = je^{j\varepsilon}$. The equation for the $\lambda_i$’s is

\[
\begin{bmatrix}
0 & -2\text{Re}[p_1\bar{p}_2] & -2\text{Re}[p_2\bar{p}_2] \\
0 & -2\text{Re}[p_1\bar{p}_1] & -2\text{Re}[p_2\bar{p}_1] \\
1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
\lambda_0 \\
\lambda_1 \\
\lambda_2
\end{bmatrix}
= 
\begin{bmatrix}
-|p_2|^2 \\
-|p_1|^2 \\
1
\end{bmatrix},
\]

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and thus
\[
\begin{bmatrix}
0 & 2\text{Re}[je^{-j\varepsilon}e^{-j\varepsilon}] & -2 \\
0 & -2 & 2\text{Re}[je^{j\varepsilon}j\varepsilon] \\
1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
\lambda_0 \\
\lambda_1 \\
\lambda_2
\end{bmatrix}
= 
\begin{bmatrix}
-1 \\
-1 \\
1
\end{bmatrix},
\]
or
\[
\begin{bmatrix}
0 & -2\text{Re}[e^{-j2\varepsilon}] & -2 \\
0 & -2 & -2\text{Re}[e^{j2\varepsilon}] \\
1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
\lambda_0 \\
\lambda_1 \\
\lambda_2
\end{bmatrix}
= 
\begin{bmatrix}
-1 \\
-1 \\
1
\end{bmatrix},
\]
or finally
\[
\begin{bmatrix}
0 & 2\cos 2\varepsilon & 2 \\
0 & 2 & 2\cos 2\varepsilon \\
1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
\lambda_0 \\
\lambda_1 \\
\lambda_2
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}.
\]
As \(\varepsilon \to 0\), the left-hand matrix becomes singular, though \(\lambda_0, \lambda_1, \lambda_2\) converge respectively to \(1/2, 1/4, 1/4\). Thus the linear equation to be solved can become ill-conditioned. A sensible thing to do if you have two points very close together (like \(p_1, p_2\)) is to ignore one of them.

Finally, consider \(n \geq 2\) point robots moving according to the circumcentre control law. Assuming the initial visibility graph has a globally reachable node, the robots rendezvous asymptotically. Thus eventually, say at time \(t_1\), they’re all visible to each other and there is only one circumcircle. On the time interval \([t_1, \infty)\), the circumcentre is stationary.

To see this, suppose three robots, \(z_1, z_2, z_3\), are on the circumcircle, all the others being inside. We have for each robot that \(\dot{z}_i = c - z_i\). Thus
\[
|\dot{z}_1| = |\dot{z}_2| = |\dot{z}_3| =: s_{\text{max}},
\]
the maximum speed, while for all other \(i\), \(|\dot{z}_i| < s_{\text{max}}\). It follows that \(z_1, z_2, z_3\) move radially toward \(c\) at the same speed and that they remain on the circumcircle as it shrinks. The centre of the circumcircle remains stationary.

### 3.6.4 Related Work

Here we look at the rendezvous problem, and extensions, in a few other contexts.

#### Discrete-Event Robots

Versions of the robot rendezvous problem have been studied extensively in computer science (where it is usually called the gathering problem). Example:

Each robot is viewed as a point in the plane. The robots have limited visibility: Each can see only the other robots within a fixed radius. Moreover, the robots are modeled as asynchronous discrete-event systems having four possible states: Wait, that is, not moving and idle; Look, during which the robot senses the relative positions of the other robots within its field of view; Compute, during which it computes its next move; and Move, during which it moves at some pre-determined speed to its computed destination. There are soft timing assumptions, such as, a robot can be in Wait for only a finite period of time.

The robots have local coordinate frames and these are assumed to have a common orientation, e.g., they may each have a compass:

![Diagram of robots with local coordinate frames](image)

The paper proposes the following control law, in the form of four if-then rules:

1. If in the Look state a robot sees a another robot to its left or vertically above, then it does not move.
2. If a robot sees robots only below on its vertical axis, then it moves down toward the nearest robot.
3. If a robot sees robots only to its right, then it moves horizontally toward the vertical axis of the nearest robot.
4. If a robot sees robots both below on its vertical axis and on its right, then it computes a certain destination point and performs a diagonal move down and to the right.

It is proved that, assuming the initial visibility graph is connected, the robots rendezvous after a finite number of events. For example, starting as in the figure before, the lower-right robot will not move, and the other three will become collocated with it. The proof is quite complicated, because, although each robot goes through a sequence of event cycles Wait-Look-Compute-Move, the robots are entirely unsynchronized, and so a robot may start to move before another has finished moving.

**Discrete-Time Robots**

Let us look in more detail at the model of Jadbabaie et al. mentioned already. For simplicity, suppose there are only two boids, each a neighbour of the other. They move at unit speed with heading angles $\theta_1, \theta_2$ with respect to the global frame. The model in is

\[
\begin{align*}
\theta_1(k+1) &= u_1(k) = \frac{1}{2} [\theta_1(k) + \theta_2(k)] \\
\theta_2(k+1) &= u_2(k) = \frac{1}{2} [\theta_1(k) + \theta_2(k)].
\end{align*}
\]
The heading angles converge in just one time step.

It may not be obvious that these steering laws are feasible by onboard sensors, but they are. Rewrite the equations as

\[
\begin{align*}
\theta_1(k+1) - \theta_1(k) &= \frac{1}{2}(\theta_2(k) - \theta_1(k)) \\
\theta_2(k+1) - \theta_2(k) &= \frac{1}{2}(\theta_1(k) - \theta_2(k)).
\end{align*}
\]

Thus the heading angles can be updated from the measured relative heading angles.

Next ref.:


Cortés et al. take the discrete-time robot model

\[
z_i(k+1) = z_i(k) + u_i(k),
\]

that is, the position update \( z_i(k+1) - z_i(k) \) is directly controllable. If only local onboard cameras are available, then again \( u_i \) must be a function of the relative positions \( z_j - z_i, j \in N_i \). The circumcentre algorithm is applied to solve the rendezvous problem.

Actually, it seems difficult to justify the above discrete-time models from our point of view of onboard sensors and distributed control. Consider, again for simplicity, just two robots, each the neighbour of the other. Suppose the robots head for each other according to the equations

\[
\begin{align*}
\dot{z}_1 &= u_1 = z_2 - z_1 \\
\dot{z}_2 &= u_2 = z_1 - z_2.
\end{align*}
\]

Now suppose the onboard controllers are digital: The sensed signals \( z_2 - z_1, z_1 - z_2 \) are sampled via a periodic sampler \( S \) with sampling period \( T \), then converted from discrete time to continuous time via a zero-order hold \( H \):

\[
\begin{array}{c}
\text{z}_2 \quad \downarrow \quad \boxed{H} \quad \uparrow \quad \boxed{u_2} \quad \downarrow \quad \boxed{z_1} \\
\text{z}_1 \quad \uparrow \quad \boxed{S} \quad \downarrow \quad \boxed{z}_2 \\
\end{array}
\]

Then the model at the sampling instants is

\[
\begin{align*}
z_1[(k+1)T] &= z_1(kT) + T[z_2(kT) - z_1(kT)] \\
z_2[(k+1)T] &= z_2(kT) + T[z_1(kT) - z_2(kT)].
\end{align*}
\]

These do have the form in (3.10), but note that it requires the two digital controllers to be synchronized somehow, by a centralized clock and communication system. So the system isn’t really distributed. Lack of synchronicity would lead to jitter, which could alternatively be modeled. (It’s interesting to note that system (3.11), (3.12) is unstable for large enough \( T \).)
### 3.6.5 Exercises

1. Consider four point robots with positions $z_1, \ldots, z_4$. The control law is that each heads for the sum of the relative distances of the others, e.g.,

$$u_1 = y_1, \quad y_1 = (z_2 - z_1) + (z_3 - z_1) + (z_4 - z_1).$$

As you know, the robots will rendezvous under this continuous-time control law. Now suppose each of the four controllers is implemented via sample and hold. For example, $y_1(t)$ is sampled to produce $y_1(kT)$ and these sampled values are passed through a zero-order hold to produce $u_1(t)$:

$$u_1(t) = y_1(kT), \quad kT \leq t < (k+1)T.$$

For what range of $T$ will the robots rendezvous? (Analytical or MATLAB solution accepted.)

2. Consider four point robots with positions $z_1, \ldots, z_4$. The control law is that each heads for the one that is farthest away from it, e.g.,

$$u_1 = m_1 - z_1, \quad m_1 = \arg\max_{j \neq 1} \|z_j - z_1\|.$$

Can you show anything about this control law? Is it Lipschitz? Does it achieve rendezvous?

3. This problem shows a different approach to the rendezvous problem, a way based on a potential function. It works only when the visibility graph is symmetric.

   (a) Consider $n$ point kinematic robots with positions $z_i$ in $\mathbb{R}^2$. Assume the visibility graph is symmetric, that is, all the links have arrows going both ways: If robot $i$ can see robot $j$, then vice versa. Assume the graph is time-invariant and connected (same as strongly connected). Let there be $m$ links in the graph.

   (b) Let $e_1, \ldots, e_m$ denote the links viewed as vectors in $\mathbb{R}^2$. That is, $e_i$ equals some $z_j - z_k$ where robots $j$ and $k$ can see each other. Writing $z$ for the aggregate state $(z_1, \ldots, z_n)$ and $e$ for the aggregate link vector $(e_1, \ldots, e_m)$, we have $e = Pz$ for a certain $2m \times 2n$ matrix $P$. (Do a little example and see what $P$ is.)

   (c) Let $L$ denote the Laplacian of the visibility graph. Thus $L$ is $n \times n$. Multiply each entry of $L$ by the $2 \times 2$ identity matrix and call the result $\hat{L}$, a $2n \times 2n$ matrix. Prove that

$$\hat{L} = P^T P. \quad (3.13)$$

(If you’re unsuccessful in a general proof, at least show it’s true for a small example, say $n = 5$.)

   (d) Now define a potential function

$$\phi = \frac{1}{2} \sum_{i=1}^{m} \|e_i\|^2 = \frac{1}{2} \|e\|^2.$$

Thus $\phi = 0$ iff the robots are all collocated; this follows because the graph is connected. This suggests a gradient control law using the Jacobian of $\phi$. 

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(e) We want to think of $\phi$ as a function of $z$, not $e$, so we write

$$\phi(z) = \frac{1}{2} \| Pz \|^2.$$ 

Let $J_{\phi}(z)$ denote the Jacobian of $\phi(z)$. Show that

$$J_{\phi}(z) = z^T \hat{L}.$$ 

(f) Finally, the control law $u = -J_{\phi}(z)^T$ results in the closed-loop system

$$\dot{z} = -\hat{L}z.$$ 

We know that rendezvous is achieved for this system.

In conclusion, we’ve derived our familiar control law $u = -\hat{L}z$ using a global potential function.

3.6.6 References

3.7 Covering an Area

So far we’ve focussed on the rendezvous problem—all robots should gather at the same location. The dual problem is for all robots to spread out as much as possible. This could be useful, for example, in deploying a team of robots to act as guards over a given space.

3.7.1 Lloyd’s algorithm in 1D

This algorithm was originally developed for the problem of quantizing data. Let \( r \) be a real number that could take any value in the interval \([0, 1]\). We want to partition \([0, 1]\) into a finite number, \( n \), of subintervals, \( \{V_i\}_{i=1}^n \), and then, for each \( i \), designate one point \( p_i \) in \( V_i \) as the codeword. Then the quantization function would be to map \( r \) to \( p_i \) if \( r \in V_i \). The partition and code book are optimal in a certain sense.

The algorithm is illustrated by an example.

**Example** \((n = 3)\) Let \( p_1 \leq p_2 \leq p_3 \) be three arbitrary points in \([0, 1]\). Construct a partition \( \{V_1, V_2, V_3\} \) as shown here:

So \( V_1 \) is from 0 to the midpoint between \( p_1 \) and \( p_2 \), \((p_1 + p_2)/2\); \( V_2 \) is from \((p_1 + p_2)/2\) to \((p_2 + p_3)/2\); and \( V_3 \) is from \((p_2 + p_3)/2\) to 1. This is called the Voronoi partition\(^2\) \( V \) generated by \( \{p_i\} \); the intervals are uniquely defined by this property: \( V_i \) is the set of all points \( q \) whose distance from \( p_i \) is less than or equal to the distances from all other \( p_j \):

Continuing with the algorithm, update \( p_i \) to be the midpoint of \( V_i \):

Then update \( V_i \) to be the Voronoi partition:

And so on. Does this procedure converge? Let \( p \) be the vector \((p_1, p_2, p_3)\). Then the update law is

\[
p(k + 1) = Ap(k) + b,
\]

\(^2\)Named after the Ukrainian mathematician Georgy Voronoi (1868–1908).
where \( b = (0, 0, 1/2) \) and
\[
A = \frac{1}{4} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}.
\]

You're invited to prove that \( p(k) \) converges to the vector \((1/6, 1/2, 5/6)\):

Thus the intervals have equal width and the points are their centres, just what you’d like for a quantizer.

---

**Continuous time**

There's a natural continuous-time version of the algorithm that will later be generalized for the robot coverage problem.

**Example** (continued) Think now of \( p_i, c_i, \) and \( V_i \) as evolving in continuous time:

\[
\dot{p}_1 = c_1 - p_1, \quad \dot{p}_2 = c_2 - p_2, \quad \dot{p}_3 = c_3 - p_3.
\]

This leads to

\[
\dot{p} = Ap + b,
\]

where \( b = (0, 0, 1/2) \) and
\[
A = \frac{1}{4} \begin{bmatrix} -3 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -3 \end{bmatrix}.
\]

Again, \( p(t) \) converges to the vector \((1/6, 1/2, 5/6)\).

---

**3.7.2 Lloyd’s algorithm in 2D**

Consider a convex polytope \( W \) in \( \mathbb{R}^2 \) of area \( A_W \). Its centroid (point of balance) \( c_W \) satisfies

\[
\int_W (q - c_W) dq = 0,
\]

and therefore

\[
c_W = \frac{1}{A_W} \int_W q dq.
\]
The polar moment of inertia of \( W \) about a point \( p \in W \) is
\[
H(p, W) = \int_W \|q - p\|^2 dq.
\]

The parallel axis theorem is a standard result in mechanics:

**Lemma 17** For every \( p \) in \( W \)
\[
H(p, W) = H(c_W, W) + A_W \|p - c_W\|^2.
\]

**Proof** The statement is
\[
\int_W \|q - p\|^2 dq = \int_W \|q - c_W\|^2 dq + \int_W \|p - c_W\|^2 dq.
\]

Now the quantity
\[
\left( \int_W \|f(q)\|^2 dq \right)^{1/2}
\]
is a norm on the function \( f : W \rightarrow \mathbb{R}^2 \), so let’s write this quantity as \( \|f\| \). Specifically, define \( f(q) = q - c_W \) (affine-linear function) and \( g(q) = c_W - p \) (constant function). Then we’re trying to show
\[
\|f + g\|^2 = \|f\|^2 + \|g\|^2,
\]
which is an instance of Pythagoras’ theorem. So all we have to show is that \( f \perp g \):
\[
\langle f, g \rangle = \int_W f(q)^T g(q) dq = \int_W (q - c_W)^T (c_W - p) dq = \int_W q^T dq(c_W - p) - \int_W dq c_W^T (c_W - p) = A_W c_W^T (c_W - p) - A_W c_W^T (c_W - p) = 0.
\]

\[\square\]

**Corollary 19** The unique point \( p \) that minimizes \( H(p, W) \) is the centroid, \( p = c_W \).
Sensor placement

Let’s interpret that last result in terms of sensor placement. Suppose we want to place a sensor at a location $p$ in $W$ to optimize coverage. We take $\mathcal{H}(p, W)$ as the cost function, a measure of coverage error—the sum over $q$ of the squares of the distances $\|q - p\|$. Then the optimal location for the sensor is the centroid.

Fixed partition

Let’s extend to $n$ sensors. Consider a convex polytope $Q$ in $\mathbb{R}^2$. Suppose $W = \{W_i\}_{i=1}^n$ is a given partition:

Now suppose there are $n$ sensors that are to be placed at locations $\{p_i\}$, one in each cell: $p_i \in W_i$:

The cost function for cell $i$ is $\mathcal{H}(p_i, W_i)$ and the total cost function is

$$\mathcal{H}(p, W) = \mathcal{H}(p_1, W_1) + \cdots + \mathcal{H}(p_n, W_n),$$

where $p = (p_1, \ldots, p_n)$ denotes the vector of sensor positions. Since $W_1, \ldots, W_n$ are all disjoint,

$$\min_p \mathcal{H}(p, W) = \min_{p_1} \mathcal{H}(p_1, W_1) + \cdots + \min_{p_n} \mathcal{H}(p_n, W_n).$$

Thus the optimal $p_i$ is the centroid of $W_i$. 

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Fixed sensor locations

Now let’s suppose the $n$ sensor locations $p$ are fixed but the $n$ cells $\mathcal{W} = \{W_i\}$ are to be designed. The optimal partition turns out to be the Voronoi partition $V$:

In mathematical terms,

$$V_i = \{q : (\forall j \neq i) \|q - p_i\| \leq \|q - p_j\|\}.$$

Each $V_i$ is the intersection of half planes. The picture just shown is called a Voronoi diagram, and the partition is uniquely determined by $p$.

Lemma 18 For a given $p$, the unique partition that minimizes $\mathcal{H}(p, \mathcal{W})$ is the Voronoi partition, $\mathcal{W} = \mathcal{V}$.

Proof Let’s do the case $n = 2$ for simplicity of explanation. Here’s the picture:

The solid line through $Q$ defines the Voronoi partition; it bisects the line joining $p_1$ and $p_2$. Let $\mathcal{W}$ be any other partition, shown by the dashed line. We’ll show that

$$\mathcal{H}(p, \mathcal{V}) \leq \mathcal{H}(p, \mathcal{W}),$$

that is,

$$\int_{V_1} \|q - p_1\|^2 dq + \int_{V_2} \|q - p_2\|^2 dq \leq \int_{W_1} \|q - p_1\|^2 dq + \int_{W_2} \|q - p_2\|^2 dq. \quad (3.14)$$
Let $\chi_V$ denote the characteristic function of a set $V$, that is,

$$\chi_V(q) = 1 \text{ if } q \in V, \quad \chi_V(q) = 0 \text{ if not.}$$

Then (3.14) is equivalent to

$$\int_Q [\|q - p_1\|^2 \chi_{V_1}(q) + \|q - p_2\|^2 \chi_{V_2}(q)] \, dq \leq$$

$$\int_Q [\|q - p_1\|^2 \chi_{W_1}(q) + \|q - p_2\|^2 \chi_{W_2}(q)] \, dq.$$

So it suffices to prove that for every $q$

$$\|q - p_1\|^2 \chi_{V_1}(q) + \|q - p_2\|^2 \chi_{V_2}(q) \leq \|q - p_1\|^2 \chi_{W_1}(q) + \|q - p_2\|^2 \chi_{W_2}(q). \quad (3.15)$$

Let $q \in V_1$. If $q \in W_1$, then

$$\|q - p_1\|^2 \chi_{V_1}(q) = \|q - p_1\|^2 \chi_{W_1}(q)$$

and so (3.15) is true with equality; whereas, if $q \in W_2$, then

$$\|q - p_1\|^2 \chi_{V_1}(q) \leq \|q - p_2\|^2 \chi_{W_2}(q)$$

and so (3.15) is true.

Likewise if $q \in V_2$.

□

Fixed number of sensors

Now we turn to the more interesting problem: Both the sensor locations and the cells are designable—only the overall set $Q$ and the number $n$ of sensors are given and fixed. The problem is to minimize $H(p, W)$ over both $p$ and $W$.

Lloyd’s algorithm is this:

Step 0: Start with an arbitrary partition $\{W_i\}$ and arbitrary points $\{p_i\}, \ p_i \in W_i$.

Step 1: Construct the unique Voronoi partition $\{V_i\}$ generated by $\{p_i\}$.

Step 2: Update $p_i$ to be the centroid of $V_i$.

Return to Step 1.

The rationale for the algorithm is this: Regarding Step 1, by Lemma 18

$$H(p, W) \geq H(p, V).$$

Then, regarding Step 2, by Lemma 17

$$H(p, V) = \sum_i \left[ H(c_{V_i}, V_i) + A_{V_i} ||p_i - c_{V_i}||^2 \right] \geq \sum_i H(c_{V_i}, V_i) = H(p_{\text{updated}}, V).$$
The procedure converges asymptotically to a Voronoi partition with \( p_i \) being the centroid of \( V_i \). However, the limit may be only a local optimum for the function \( \mathcal{H} \). For example:

If the algorithm is initialized in either of the two ways shown, it terminates immediately. But the right-hand value of \( \mathcal{H} \) is larger than the left-hand value.

### 3.7.3 Robot Coverage

Consider \( n \) mobile robots moving in the plane. Take the usual kinematic model

\[
\dot{p}_i = u_i.
\]

The goal is for the robots to deploy themselves to provide adequate coverage of a given convex polytope \( Q \).

For each time \( t \), let \( \mathcal{V}(t) \) denote the Voronoi partition generated by the points

\[
p(t) = (p_1(t), \ldots, p_n(t)).
\]

Also, let \( c_i(t) \) denote the centroid of \( V_i(t) \) and define the vector

\[
c_V(t) = (c_1(t), \ldots, c_n(t)).
\]

Based on Lloyd’s algorithm, Cortes et al. propose the control strategy that robot \( i \) should head for the centroid of its Voronoi cell:

\[
u_i = c_i - p_i.
\]

The aggregate vector \( p \) is governed by

\[
\dot{p} = c_V - p.
\]

The block diagram for this model is
Of course, this could not be done in real time and a sampled-data implementation would be required:

The algorithm for two robots covering a square is illustrated in this picture:

3.7.4 Convergence

We return to the 1D continuous-time Lloyd algorithm and look at convergence from the viewpoint of Lyapunov theory.

Let’s do an even simpler example: 2 robots.

Example Let \( p_1 \leq p_2 \) be two arbitrary points in \([0, 1]\). Construct the Voronoi partition \( \{V_1, V_2\} \):

Let \( a \) denote the midpoint between \( p_1 \) and \( p_2 \), \( (p_1 + p_2)/2 \); so

\[
V_1 = [0, a], \quad V_2 = [a, 1].
\]
Also, let $c_1, c_2$ denote the midpoints of $V_1, V_2$. Finally, let the points evolve according to

$$\dot{p}_1 = c_1 - p_1, \quad \dot{p}_2 = c_2 - p_2.$$  

Notice that $a, c_1, c_2$ are all functions of $p_1, p_2$:

$$c_1 = \frac{p_1 + p_2}{4}, \quad a = \frac{p_1 + p_2}{2}, \quad c_2 = \frac{p_1 + p_2}{4} + \frac{1}{2}.$$  

Define

$$L(p_1, p_2) = \int_0^a (q - p_1)^2 dq + \int_1^a (q - p_2)^2 dq.$$  

From the parallel axis theorem,

$$L(p_1, p_2) = \int_0^a (q - c_1)^2 dq + \int_1^a (q - c_2)^2 dq + a(p_1 - c_1)^2 + (1 - a)(p_2 - c_2)^2. \quad (3.16)$$  

To use this as a Lyapunov function, we need to study

$$\frac{d}{dt} L(p_1(t), p_2(t)) = \sum_{i=1,2} \frac{\partial L}{\partial p_i}(p_1(t), p_2(t)) \dot{p}_i(t). \quad (3.17)$$  

In particular, the partial derivative $\partial L/\partial p_1$ would seem to have eight terms. In fact seven cancel, and

$$\frac{\partial L}{\partial p_1} = 2a(p_1 - c_1), \quad (3.18)$$  

which is what one would get from (3.16) by treating $a, c_1, c_2$ as constants. Equation (3.18) can be verified by MATLAB symbolic computation:

```matlab
syms p1 p2 q c1 c2 real;
  a = (p1 + p2)/2;
  H = int((q-p1)^2,q,0,a) + int((q-p2)^2,q,a,1);
  dh = jacobian(H,[p1 p2]);
  c1 = (p1+p2)/4;
  expand(dh(1) - 2*a*(p1-c1))
  ans =
        0
```

We omit the mathematical proof of (3.18).

Using (3.18) in (3.17), we get

$$\frac{d}{dt} L = 2a(p_1 - c_1)\dot{p}_1 + 2(1 - a)(p_2 - c_2)\dot{p}_2$$

$$= -2a(p_1 - c_1)^2 - 2(1 - a)(p_2 - c_2)^2 \leq 0.$$  

One can then argue that the robots converge to a local optimum of $L$ where $p_i = c_i$.  

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3.7.5 Exercises

1. This is a problem on sensor placement. Suppose we want to place a sensor to detect, say, temperature.

   (a) Suppose the workspace is the unit interval $[0, 1]$. We want to place a sensor at a location $p \in [0, 1]$ to get “optimal coverage.” To make this precise, we have to define a measure of coverage error, denoted $H(p)$. Here are some options:

   If $q$ is another point, how well the sensor can measure the temperature at location $q$ depends on the distance $|q - p|$. Suppose the temperature error is in fact proportional to $|q - p|$. Then the average error is proportional to

   $$H_1(p) = \int_0^1 |q - p| dq$$

   while the worst case error is proportional to

   $$H_\infty(p) = \max_{0 \leq q \leq 1} |q - p|.$$

   Suppose the temperature error is in fact proportional to $|q - p|^2$. Then the average error is proportional to

   $$H_2(p) = \int_0^1 |q - p|^2 dq.$$

   Find the optimal $p$ (the one that minimizes $H$) in each of the three cases. Is the optimal $p$ the centre of the interval in all three cases?

   (b) Now consider the extension to a convex polygon region in $\mathbb{R}^2$. As in the course notes, for $H_2(p)$, the optimal $p$ is the centroid of the region. Give an example where this isn’t true for $H_1(p)$ or $H_\infty(p)$.

3.7.6 References


3.8 Discrete Robots

Our robots are dynamical systems with differential equation models. There’s a literature on robot formations where the robots are modeled as finite state machines. We’ll call these “discrete robots.” In this section we look at one solution of the rendezvous problem for discrete robots. Moreover, the robots are asynchronous—they do not have a common clock. To balance this generality, the robots do know a common direction; perhaps they all have compasses, or can all see a very remote beacon.

3.8.1 The Setup

1. The robots are points in $\mathbb{R}^2$.

2. Each robot has a local $(x, y)$-coordinate frame. We assume all the $x$-axes are parallel, and for convenience, parallel to the global $x$-axis. Like this:

   ![Diagram of robot movements](image)

   It is not assumed, however, that they have a common measure of unit length.

3. The robots do not have a common clock.

4. The robots are indistinguishable.

5. The robots have limited visibility: They can see at most a fixed distance $d > 0$ away.

6. Now for the finite state part. Each robot can be in one of four states:

   ![Finite state diagram](image)

   The states are Wait or idle; Look, where the robot takes a snapshot of the other robots within distance $d$; Compute, where the next desired destination point (new or the same) is computed; and Move, where the robot moves at a fixed speed toward the computed destination, though
it may terminate (and go to Wait) before it arrives. It is assumed that there are upper and lower time bounds during which each robot can be in any of the four states (e.g., a robot can’t Wait forever).

3.8.2 The Control Law

At global time $0$, the robots are in a minimal properly oriented rectangle:

\[
\text{\begin{tikzpicture}
  \draw[->] (0,0) -- (5,0);
  \draw[->] (0,0) -- (0,5);
  \filldraw[black] (1,1) circle (2pt);
  \filldraw[black] (2,2) circle (2pt);
  \filldraw[black] (3,3) circle (2pt);
  \filldraw[black] (4,4) circle (2pt);
  \filldraw[black] (1,4) circle (2pt);
  \filldraw[black] (1,3) circle (2pt);
  \filldraw[black] (3,1) circle (2pt);
  \filldraw[black] (4,2) circle (2pt);
\end{tikzpicture}}
\]

This rectangle will be an invariant set, and the rendezvous point will be the lower-right corner.

Let $r$ be a generic robot. In its Compute state it determines its next destination, $r'$, based on the robots that were visible when it was last in its Look state. This computation goes as follows:

1. If $r$ sees robots to its left or vertically above, then $r' = r$ (it doesn’t move).
2. If $r$ sees robots only vertically below, then $r'$ equals the nearest.
3. If $r$ sees robots only to its right, let $r_1$ be the one such that
   \[
   |\text{proj}_x(r_1 - r)| = \text{minimum}.
   \]
   Then $r'$ is the intersection of the horizontal line through $r$ and the vertical line through $r_1$:

\[
\text{\begin{tikzpicture}
  \draw[->] (0,0) -- (5,0);
  \draw[->] (0,0) -- (0,5);
  \filldraw[black] (1,1) circle (2pt);
  \filldraw[black] (2,2) circle (2pt);
  \filldraw[black] (3,3) circle (2pt);
  \filldraw[black] (4,4) circle (2pt);
  \filldraw[black] (2,1) circle (2pt);
  \filldraw[black] (3,2) circle (2pt);
  \filldraw[black] (4,3) circle (2pt);
  \filldraw[black] (2,3) circle (2pt);
  \filldraw[black] (3,2) circle (2pt);
  \filldraw[black] (4,3) circle (2pt);
  \filldraw[black] (3,1) circle (2pt);
  \filldraw[black] (4,2) circle (2pt);
  \filldraw[black] (1,4) circle (2pt);
  \filldraw[black] (1,3) circle (2pt);
  \filldraw[black] (3,1) circle (2pt);
  \filldraw[black] (4,2) circle (2pt);
  \draw[dashed] (1,1) -- (4,4);
  \draw[dashed] (1,1) -- (1,4);
  \draw[dashed] (2,2) -- (4,2);
  \draw[dashed] (2,2) -- (3,1);
\end{tikzpicture}}
\]

4. Finally, if $r$ sees robots both vertically below and to its right, let $r_1$ be the closest one vertically below and let $r_2$ be the one to the right such that
   \[
   |\text{proj}_x(r_2 - r)| = \text{minimum}.
   \]
   Then $r'$ is computed from the following picture:
That is, one draws the two vertical lines through $r_1, r_2$ and also the circle of radius $d$. Then the parallelogram determines $r'$, down-and-right of $r$.

Actually, there’s a problem if the angle $\alpha$ is less than $2\pi/3$, for then $r'$ is outside the disk of visibility:

In this case, in the computation one translates $r_2$ left just until $r'$ meets the circle.

**Theorem 20** If the visibility graph is initially connected, the robots rendezvous in finite time.

The proof is very involved because of all the possibilities. For example, after one robot has taken a snapshot but before its next move has completed, other robots may have moved.

### 3.8.3 References

3.9 Robots on a Grid

In this very brief section we look at a possible research problem motivated by the subject of cellular automata. A cellular automaton is a discrete model studied in computability theory, mathematics, and theoretical biology.

Consider $N$ robots moving on the grid $\mathbb{Z}^2$:

![Grid with robots](image)

The problem is to find a distributed control law so the robots rendezvous.

3.9.1 First try

The setup:

1. Each robot can see only the four adjacent nodes (North, South, East, West).

2. Each robot may move one step according to what it sees.

3. If two robots come to be co-located, they forever after move as one.

4. The robots form a graph: There’s a link between two robots iff they’re adjacent. The graph is initially connected.

5. The robots’ clocks are unsynchronized, but no two robots can move simultaneously (a simplification). So every once in a while, a robot moves.

Suppose the initial graph is a tree, as above. Then a very simple control law to get rendezvous is this: If you have only one neighbour, move to that node. That is, only leaves can move. If there exists a $T > 0$ such that in every time interval of width $T$ there’s at least one move, then the robots rendezvous.

If the graph is not a tree, the problem is harder, as is shown by this example:

**Example** Consider these two examples:
In the left-hand case, $r_1$ may move, indeed, the loop must somehow be broken. In the right-hand case, $r_1$ must not move, for then the graph becomes disconnected. But $r_1$ has the same local view in each case. This suggests the problem isn’t solvable.

3.9.2 Exercises

1. How should the problem be changed so it is solvable? What’s a solution?

2. This problem involves the *peg game* or *solitary checkers*. Consider an infinite checkerboard. At time 0, a finite number of checkers are in a certain pattern on the board. At time $k$ a checker can move by jumping an adjacent one to the right, left, up, or down. The jumped checker is removed from the board. The game is to get down to only one checker.

A variation is where the checkerboard has a horizontal line and at time 0 all the checkers are below the line. How far above the line can some checker get? For example, here’s a game with 8 initial checkers and where one gets 3 squares above the line (the red checker is the next to move):

![Checkerboard diagram](image)

The mathematician John Conway, who invented the Game of Life, proved that there is no initial pattern so that some checker will get 5 squares across the line. The ingenious proof is as follows.

(a) Suppose, for a proof by contraction, there’s an initial pattern of a finite number of checkers below the line so that some checker ends up 5 squares above the line after a finite sequence of checker moves.
(b) Let $s$ be the positive root of the equation $s^2 + s = 1$. Assign the weight 1 to the terminal square (5 above the line) and the weight $s^m$ to every square that is $m$ squares from the terminal square:

\[
\begin{array}{ccc}
  s & 1 & s \\
  s^2 & s & s^2 \\
  s^3 & s^2 & s^3 \\
  s^4 & s^3 & s^4 \\
  s^5 & s^4 & s^5 \\
  s^6 & s^5 & s^6 \\
\end{array}
\]

(c) Prove that the sum of all the weights below the line equals 1.

(d) Let $k$ be a generic time and let $W(k)$ denote the sum of the weights of all squares that have checkers on them at time $k$. From the preceding step, $W(0) < 1$.

(e) Prove that $W(k + 1) \leq W(k)$, i.e., no checker move can increase $W(k)$.

(f) At the terminal time, $W(k) \geq 1$. Contradiction.

Study the preceding argument for the case of getting 3 squares above the line. Show that there’s no contradiction.

3.9.3 References


2. The literature on cellular automata.

3. The Game of Life.

4. Wolfram’s book on cellular automata

3.9.4 Appendix

// A Scilab program to study robots on a grid.

clear;
// A is a rectangular matrix of 0’s and 1’s. A 1 indicates a robot location.
// Load A. Choose a specific A or a random A.
// random:
nr=5; // number of rows
nc=5; // number of columns
A=rand(nr,nc);
A=round(A);
// specific:
A=
    [0 0 1 0 0 0 0 1 0 0 0 0 0 1 0 1 0 1 0 1 0 1]

[nr,nc]=size(A);
// It’s convenient to pad with zeros.
A=[0 zeros(1,nc) 0;zeros(nr,1) A zeros(nr,1); 0 zeros(1,nc) 0];

[nr,nc]=size(A);
// Two robots are neighbours if they’re adjacent on the grid.
// Calculate the number of nodes, n.
n=sum(A);
// Order the nodes in raster fashion, left-to-right, top-to-bottom.
// Build the matrix k of nodes.
k=0*A;
um=0;
for i=1:nr;
for j=1:nc;
if A(i,j)==1 then num=num+1;
else irrel=0;
end
if A(i,j)==1 then k(i,j)=num;
else irrel=0;
end
end
end
// Now k represents the graph. Next step is to check if it’s connected.
// Form the adjacency matrix, AA.
AA=zeros(n,n);
for i=1:nr;
for j=1:(nc-1);
if (k(i,j)<k(i,j+1)) & (k(i,j)>0) then AA(k(i,j),k(i,j+1))=1;
else irrel=0;
end
end
for i=1:(nr-1);
for j=1:nc;
if (k(i,j)<k(i+1,j)) & (k(i,j)>0) then AA(k(i,j),k(i+1,j))=1;
else irrel=0;
end
end
end
AA=AA+AA';
// Form the degree matrix, D.
un=ones(n,1); // vector of ones
D=diag(AA*un);
// Form the Laplacian matrix, L.
L=D-AA;
test=n-rank(L); // connected iff test=1
if test==1 then irrel=0;
else return;
end
// Now for the robot moves.
// Any leaf can move.
// Suppose a robot at (i,j) has only one neighbour. Then
// it can move to that neighbour.
// while sum(A)>1
for k=1:1000;
i=grand(1,1,'uin',2,nr-1);
j=grand(1,1,'uin',2,nc-1);
num_neighbours=sum(A([i-1 i+1],j))+sum(A(i,[j-1 j+1]));
if num_neighbours==1 then A(i,j)=0;
else irrel=0;
end
end
return
Chapter 4

The Unicycle

4.1 Introduction

We choose a unicycle as our robot of study. We study a unicycle because it’s the simplest model of a wheeled vehicle.

A schematic:

Equivalent system: shopping cart
The shopping cart has two independent motor drives at the rear wheels. The centre of mass is at centre of the axle.

The unicycle or shopping cart can move only in the direction it’s heading. That is, there’s a no-slip condition:

\[
\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \perp \begin{bmatrix} \sin(\theta) \\ -\cos(\theta) \end{bmatrix}
\]

i.e.,

\[
\dot{x} \sin(\theta) - \dot{y} \cos(\theta) = 0
\]

This is called a **nonholonomic constraint**.

More generally, consider a mechanical system with position vector \( q \) and velocity vector \( \dot{q} \). A position constraint of the form

\[
h(q) = 0
\]

is called a **holonomic** constraint. Example:
Let the positions of the masses be $q_1, q_2 \in \mathbb{R}^3$. Then the holonomic constraint is

$$\|q_1 - q_2\|^2 - L = 0.$$ 

Thus a holonomic constraint restricts the motion to a surface in position space and reachability in configuration space is not possible. 

A constraint of the form

$$h(q) \dot{q} = 0$$

that cannot be integrated into a position constraint is called a nonholonomic constraint. Example, unicycle:

$$\dot{x} \sin(\theta) - \dot{y} \cos(\theta) = \begin{bmatrix} \sin(\theta) & -\cos(\theta) & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = 0.$$ 

Reachability may still be possible; e.g., a unicycle can be parked in any configuration:
4.1.1 State Model

Kinematic model (no dynamics; no mass, inertia):

\[
\begin{align*}
\dot{x} &= v \cos(\theta) \\
\dot{y} &= v \sin(\theta) \\
\dot{\theta} &= \omega
\end{align*}
\]

Configuration vector: \((x, y, \theta)\). Input vector: \((v, \omega)\).

The form is

\[
\dot{q} = f(q, u)
\]

where

\[
f(q, u) = f(x, y, \theta; v, \omega) = \begin{bmatrix} v \cos(\theta) \\ v \sin(\theta) \\ \omega \end{bmatrix},
\]

or

\[
\dot{q} = v g_1(q) + \omega g_2(q) = \begin{bmatrix} g_1(q) & g_2(q) \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix},
\]

where

\[
g_1(q) = f(x, y, \theta; 1, 0) = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{bmatrix}, \quad g_2(q) = f(x, y, \theta; 0, 1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
\]

Thus the model is linear in the input.

The unicycle model is

\[
\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = v \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{bmatrix} + \omega \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
\]

This is of the form

\[
\dot{q} = g_0(q) + u_1 g_1(q) + \cdots + u_m g_m(q),
\]
which is called an **affine control system**, since for each $q$ the mapping
\[
u = (u_1, \ldots, u_m) \mapsto g_0(q) + u_1 g_1(q) + \cdots + u_m g_m(q) : \mathbb{R}^m \rightarrow \mathbb{R}^n
\]
is an affine function (translate of a linear function). The control-less term $g_0(q)$ is called the **drift**. Thus the unicycle is drift-free: There’s no motion when the inputs are zero. A unicycle on a moving flatbed truck would have a drift term.

### 4.1.2 Controllability

Here we introduce some geometric control theory and see that the unicycle is a controllable nonlinear system. References:


Let $\mathcal{F}(\mathbb{R}^n)$ denote the space of all smooth functions $\mathbb{R}^n \rightarrow \mathbb{R}^n$. E.g., unicycle:

\[
q = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}, \quad g_1(q) = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{bmatrix}, \quad g_2(q) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad g_1, g_2 \in \mathcal{F}(\mathbb{R}^3).
\]

The space $\mathcal{F}(\mathbb{R}^n)$ is a vector space over the field $\mathbb{R}$; it’s infinite-dimensional since it doesn’t have a finite basis. For $f, g \in \mathcal{F}(\mathbb{R}^n)$, their **Lie bracket** (pronounced *Lee*) is defined to be

\[
[f, g](q) = \partial g \bigg|_{q} \frac{\partial f}{\partial q} - \partial f \bigg|_{q} \frac{\partial g}{\partial q}.
\]

E.g., unicycle:

\[
\frac{\partial g_1}{\partial q} = \begin{bmatrix} 0 & 0 & -\sin(\theta) \\ 0 & 0 & \cos(\theta) \\ 0 & 0 & 0 \end{bmatrix}, \quad \frac{\partial g_2}{\partial q} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

\[
[g_1, g_2](q) = -\frac{\partial g_1}{\partial x} g_2(x) = \begin{bmatrix} \sin(\theta) \\ -\cos(\theta) \\ 0 \end{bmatrix}
\]

Thus $[f, g] \in \mathcal{F}(\mathbb{R}^n)$ and so $\mathcal{F}(\mathbb{R}^n)$ is closed under Lie bracket. So $\mathcal{F}(\mathbb{R}^n)$ is a vector space and also it has a “product” operation defined on it; this makes it an **algebra**. Because the product operation is taking Lie bracket, we say $\mathcal{F}(\mathbb{R}^n)$ is a **Lie algebra**.

Returning to the unicycle, consider the two vectors $\{g_1, g_2\}$ in $\mathcal{F}(\mathbb{R}^3)$. Their span over $\mathbb{R}$, that is, the set of all linear combinations

\[c_1 g_1 + c_2 g_2, \quad c_1, c_2 \in \mathbb{R},\]
denoted \( \text{span}_\mathbb{R}\{g_1, g_2\} \), is a two-dimensional subspace of the infinite-dimensional vector space \( \mathcal{F}(\mathbb{R}^3) \). However, \( \text{span}_\mathbb{R}\{g_1, g_2\} \) is not a subalgebra, because it is not closed under Lie bracket since \([g_1, g_2] \notin \text{span}_\mathbb{R}\{g_1, g_2\} \):

\[
\begin{bmatrix}
\sin(\theta) \\
-\cos(\theta) \\
0
\end{bmatrix} \neq c_1 \begin{bmatrix}
\cos(\theta) \\
\sin(\theta) \\
0
\end{bmatrix} + c_2 \begin{bmatrix} 0 \\
0 \\
1
\end{bmatrix}.
\]

Let us add this extra function to get \( \text{span}_\mathbb{R}\{g_1, g_2, [g_1, g_2]\} \), a three-dimensional vector subspace of \( \mathcal{F}(\mathbb{R}^3) \). It is closed under Lie bracket:

\([g_1, [g_1, g_2]] = 0, [g_2, [g_1, g_2]] = g_1 \).

It is therefore a Lie algebra itself. We denote \( \text{span}_\mathbb{R}\{g_1, g_2, [g_1, g_2]\} \) by Lie\(\{g_1, g_2\}\), the **Lie algebra generated by** \( \{g_1, g_2\} \).

More generally, let \( S \) be a subset of \( \mathcal{F}(\mathbb{R}^n) \). E.g., \( S = \{g_1, \ldots, g_m\} \), a finite set. The Lie algebra generated by \( S \), denoted Lie\(\{S\}\), is the smallest vector subspace of \( \mathcal{F}(\mathbb{R}^n) \) with the two properties: It contains \( S \) and it is closed under Lie bracket, i.e.,

\[(\forall f, g \in \text{Lie}(S))[f, g] \in \text{Lie}(S).\]

For each fixed \( q \in \mathbb{R}^n \) and \( f \in \mathcal{F}(\mathbb{R}^n) \), \( f(q) \) is a vector in \( \mathbb{R}^n \). Again, for \( S \) a subset of \( \mathcal{F}(\mathbb{R}^n) \), define

\[
\text{Lie}_q(S) = \{ f(q) : f \in \text{Lie}(S) \}.
\]

Thus \( \text{Lie}_q(S) \) is a vector subspace of \( \mathbb{R}^n \).

For the unicycle

\[
g_1(q) = \begin{bmatrix}
\cos(\theta) \\
\sin(\theta) \\
0
\end{bmatrix}, \quad g_2(q) = \begin{bmatrix} 0 \\
0 \\
1
\end{bmatrix}, \quad [g_1, g_2](q) = \begin{bmatrix}
\sin(\theta) \\
-\cos(\theta) \\
0
\end{bmatrix}
\]

\[
\text{Lie}\{g_1, g_2\} = \text{span}_\mathbb{R}\{g_1, g_2, [g_1, g_2]\}
\]

\[
\text{Lie}_q\{g_1, g_2\} = \{ f(q) : f \in \text{Lie}\{g_1, g_2\} \} = \text{span}_\mathbb{R}\{g_1(q), g_2(q), [g_1, g_2](q)\} = \mathbb{R}^3, \forall q.
\]

**Example** This is an interesting linear example. Let \( b \in \mathbb{R}^n \), \( A \in \mathbb{R}^{n \times n} \), and define

\[
f(x) = b, g(x) = Ax.
\]

Of course, these are smooth, so \( f, g \in \mathcal{F}(\mathbb{R}^n) \); \( f \) is a constant function, \( g \) is a linear function. What is the Lie algebra, \( \text{Lie}\{f, g\} \), generated by the two functions? Note that

\[
[f, g](x) = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x) = Ab
\]

\[
[[f, g], g](x) = A^2 b
\]
and so on. It follows that \( \text{Lie}(\{f, g\}) \) equals the span over \( \mathbb{R} \) of the functions 
\[
Ax, b, Ab, \ldots, A^{n-1}b.
\]
(Terminates by Cayley-Hamilton.) Thus \( \text{Lie}(\{f, g\}) \) is a finite-dimensional subspace of \( \mathcal{F}(\mathbb{R}^n) \). For \( x = 0 \), \( \text{Lie}_x(\{f, g\}) \) is the finite-dimensional subspace
\[
\text{span}_\mathbb{R}\{b, Ab, \ldots, A^{n-1}b\}
\]
of \( \mathbb{R}^n \), the controllable subspace of the pair \((A, b)\).

Why the Lie bracket is important in nonlinear controllability

Consider the 2-input system, like the unicycle:
\[
\dot{q} = u_1 g_1(q) + u_2 g_2(q).
\]
Starting at \( q_0 = q(0) \), what states are reachable after a brief time interval? By taking 
\[
u_1(t) = 1, \quad u_2(t) = 0
\]
we can head off in the direction \( g_1(q_0) \). Likewise in the direction \( g_2(q_0) \). What other directions? Let us consider in particular the following control action:

Where is the state at \( t = 4\varepsilon \)? Can use Taylor’s series to answer.

**Template** \( \dot{q} = f(q), \) time interval \([t_0, t_0 + \varepsilon]\):
\[
\dot{q} = f \implies \ddot{q} = \frac{\partial f}{\partial q} \dot{q} = \frac{\partial f}{\partial q} f
\]
\[
q(t_0 + \varepsilon) = q(t_0) + \varepsilon \dot{q}(t_0) + \frac{1}{2} \varepsilon^2 \ddot{q}(t_0) + O(\varepsilon^3)
\]
\[
= q(t_0) + \varepsilon f(q(t_0)) + \frac{1}{2} \varepsilon^2 \frac{\partial f}{\partial q}(q(t_0)) f(q(t_0)) + O(\varepsilon^3)
\]
In the following derivation, it’s convenient to drop the evaluation argument $q_0$, like this:

**Step 1** $\dot{q} = g_1(q)$, time interval $[0, \varepsilon)$:

\[
q(\varepsilon) = q_0 + \varepsilon \dot{q}(0) + \frac{1}{2} \varepsilon^2 \ddot{q}(0) + O(\varepsilon^3)
\]

\[
= q_0 + \varepsilon g_1(q_0) + \frac{1}{2} \varepsilon^2 \frac{\partial g_1}{\partial q}(q_0)g_1(q_0) + O(\varepsilon^3)
\]

\[
= q_0 + \varepsilon g_1 + \frac{1}{2} \varepsilon^2 \frac{\partial g_1}{\partial q} g_1 + O(\varepsilon^3)
\]

**Step 2** $\dot{q} = g_2(q)$, time interval $[\varepsilon, 2\varepsilon)$:

\[
q(2\varepsilon) = q(\varepsilon) + \varepsilon \dot{q}(\varepsilon) + \frac{1}{2} \varepsilon^2 \ddot{q}(\varepsilon) + O(\varepsilon^3)
\]

\[
= q(\varepsilon) + \varepsilon g_2(q(\varepsilon)) + \frac{1}{2} \varepsilon^2 \frac{\partial g_2}{\partial q}(q(\varepsilon))g_2(q(\varepsilon)) + O(\varepsilon^3)
\]

But from **Step 1**

\[
q(\varepsilon) = q_0 + \varepsilon g_1 + \frac{1}{2} \varepsilon^2 \frac{\partial g_1}{\partial q} g_1 + O(\varepsilon^3),
\]

so

\[
g_2(q(\varepsilon)) = g_2(q_0 + \varepsilon g_1 + O(\varepsilon^3))
\]

\[
= g_2 + \varepsilon \frac{\partial g_2}{\partial q} g_1 + O(\varepsilon^2)
\]

and

\[
\frac{\partial g_2}{\partial q}(q(\varepsilon))g_2(q(\varepsilon)) = \frac{\partial g_2}{\partial q} g_2 + O(\varepsilon).
\]

Thus

\[
q(2\varepsilon) = \left\{ q_0 + \varepsilon g_1 + \frac{1}{2} \varepsilon^2 \frac{\partial g_1}{\partial q} g_1 \right\} + \varepsilon \left\{ g_2 + \varepsilon \frac{\partial g_2}{\partial q} g_1 \right\} + \frac{1}{2} \varepsilon^2 \frac{\partial g_2}{\partial q} g_2 + O(\varepsilon^3)
\]

\[
= q_0 + \varepsilon \{ g_1 + g_2 \} + \frac{1}{2} \varepsilon^2 \left\{ \frac{\partial g_1}{\partial q} g_1 + 2 \frac{\partial g_2}{\partial q} g_1 + \frac{\partial g_2}{\partial q} g_2 \right\} + O(\varepsilon^3).
\]

**Step 3** $\dot{q} = -g_1(q)$, time interval $[2\varepsilon, 3\varepsilon)$:

\[
q(3\varepsilon) = q(2\varepsilon) + \varepsilon \dot{q}(2\varepsilon) + \frac{1}{2} \varepsilon^2 \ddot{q}(2\varepsilon) + O(\varepsilon^3)
\]

\[
= q(2\varepsilon) - \varepsilon g_1(q(2\varepsilon)) + \frac{1}{2} \varepsilon^2 \frac{\partial g_1}{\partial q}(q(2\varepsilon))g_1(q(2\varepsilon)) + O(\varepsilon^3)
\]

But from **Step 2**

\[
q(2\varepsilon) = q_0 + \varepsilon \{ g_1 + g_2 \} + \frac{1}{2} \varepsilon^2 \left\{ \frac{\partial g_1}{\partial q} g_1 + 2 \frac{\partial g_2}{\partial q} g_1 + \frac{\partial g_2}{\partial q} g_2 \right\} + O(\varepsilon^3),
\]

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\[
g_1(q(2\varepsilon)) = g_1(q_0 + \varepsilon g_1 + g_2) + O(\varepsilon^2) \\
= g_1 + \varepsilon \frac{\partial g_1}{\partial q}(g_1 + g_2) + O(\varepsilon^2)
\]

and
\[
\frac{\partial g_1}{\partial q}(q(2\varepsilon)) g_1(q(2\varepsilon)) = \frac{\partial g_1}{\partial q}(g_1 + g_2) + O(\varepsilon).
\]

After canceling terms
\[
q(3\varepsilon) = q_0 + \varepsilon g_2 + \frac{1}{2} \varepsilon^2 \left\{ \frac{\partial g_2}{\partial q} g_2 + 2 \frac{\partial g_2}{\partial q} g_1 - 2 \frac{\partial g_1}{\partial q} g_2 \right\} + O(\varepsilon^3).
\]

**Step 4** \( \dot{q} = -g_2(q) \), time interval \([3\varepsilon, 4\varepsilon)\):

\[
q(4\varepsilon) = q(3\varepsilon) + \varepsilon \dot{q}(3\varepsilon) + \frac{1}{2} \varepsilon^2 \ddot{q}(3\varepsilon) + O(\varepsilon^3) \\
= q(3\varepsilon) - \varepsilon g_2(q(3\varepsilon)) + \frac{1}{2} \varepsilon^2 \frac{\partial g_2}{\partial q}(q(3\varepsilon)) g_2(q(3\varepsilon)) + O(\varepsilon^3)
\]

But from Step 3
\[
q(3\varepsilon) = q_0 + \varepsilon g_2 + \frac{1}{2} \varepsilon^2 \left\{ \frac{\partial g_2}{\partial q} g_2 + 2 \frac{\partial g_2}{\partial q} g_1 - 2 \frac{\partial g_1}{\partial q} g_2 \right\} + O(\varepsilon^3),
\]

so
\[
g_2(q(3\varepsilon)) = g_2(q_0 + \varepsilon g_2 + O(\varepsilon^2)) \\
= g_2 + \varepsilon \frac{\partial g_2}{\partial q} g_2 + O(\varepsilon^2)
\]

and
\[
\frac{\partial g_2}{\partial q}(q(3\varepsilon)) g_2(q(3\varepsilon)) = \frac{\partial g_2}{\partial q} g_2 + O(\varepsilon).
\]

After canceling terms
\[
q(4\varepsilon) = q_0 + \varepsilon^2 \left\{ \frac{\partial g_2}{\partial q} g_1 - \frac{\partial g_1}{\partial q} g_2 \right\} + O(\varepsilon^3) = q_0 + \varepsilon^2 [g_1, g_2] + O(\varepsilon^3).
\]

**Recap**

\[
\dot{q} = u_1 g_1(q) + u_2 g_2(q), \quad q(0) = q_0
\]
\[ q(4\varepsilon) = q_0 + \varepsilon^2 [g_1, g_2] + O(\varepsilon^3) \]

**Conclusion** Starting at \( q_0 \), we can instantaneously drive the state \( q \) not just in the directions \( g_1(q_0) \) and \( g_2(q_0) \), but also in the direction \([g_1, g_2](q_0)\). Likewise also in the direction \([g_1, [g_1, g_2]](q_0)\), etc. Indeed, in any direction in \( \text{Lie}_{q_0}([g_1, g_2]) \). Therefore, intuitively, the system

\[
\dot{q} = u_1 g_1(q) + u_2 g_2(q)
\]

is controllable at the point \( q \) if

\[
\text{dim Lie}_q([g_1, g_2]) = \text{dim } q.
\]

The mathematical statement of this fact is known as Chow’s theorem (1939).

### 4.1.3 Frenet-Serret Frame

The Frenet-Serret frame is a moving frame attached to a vehicle. We can identify the real plane, \( \mathbb{R}^2 \), and the complex plane, \( \mathbb{C} \), by identifying a column vector, \( z \), and a complex number, \( z \).

Consider a unicycle moving at unit forward speed and with coordinates \((x, y, \theta)\) with respect to a global frame. The location of the unicycle in the plane is

\[
z = \begin{bmatrix} x \\ y \end{bmatrix} \text{ or } z = x + jy.
\]

The kinematic equations are

\[
\begin{align*}
\dot{x} &= \cos(\theta) \\
\dot{y} &= \sin(\theta) \\
\dot{\theta} &= \omega
\end{align*}
\]

or

\[
\begin{align*}
\dot{z} &= e^{j\theta} \\
\dot{\theta} &= \omega
\end{align*}
\]

We now want to construct a moving frame that is fixed to the unicycle.
Let \( \mathbf{r} \) be the unit vector tangent to the trajectory at the current location of the unicycle and in the direction of motion, and let \( \mathbf{s} \) be \( \mathbf{r} \) rotated by \( \pi/2 \). Since the unicycle is moving at unit speed, \( \mathbf{r} = \dot{\mathbf{z}}, \) and so
\[
\mathbf{r} = \dot{\mathbf{z}} = e^{i\theta}, \quad \mathbf{s} = j\mathbf{r}.
\]
Thus
\[
\dot{\mathbf{r}} = \frac{d}{dt} e^{i\theta} = j e^{i\theta} \dot{\theta} = j r \dot{\theta} = \mathbf{s} \omega
\]
and
\[
\dot{s} = j \dot{r} = js \omega = -r \omega.
\]

Summary

The kinematic equations using the Frenet-Serret frame are
\[
\begin{align*}
\dot{\mathbf{z}} & = \mathbf{r} \\
\dot{\mathbf{r}} & = \mathbf{s} \omega \\
\dot{s} & = -r \omega.
\end{align*}
\]

4.1.4 Finally

Finally, our unicycle is kinematic. The simplest dynamic model is
\[
\begin{align*}
\dot{x} & = v \cos(\theta) \\
\dot{y} & = v \sin(\theta) \\
\dot{\theta} & = \omega \\
\dot{v} & = \frac{1}{M} f \text{ (Newton)} \\
\dot{\omega} & = \frac{1}{I} \tau \text{ (Newton)}
\end{align*}
\]
Five states, two force inputs.
4.1.5 Exercises

1. Let $M$ be an $n \times m$ matrix and $b$ an $n \times 1$ vector. Consider the affine function $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ given by $f(u) = Mu + b$. Describe what kind of geometric objects are the image

$$\{ y : y = f(u), u \in \mathbb{R}^m \}$$

and the graph

$$\{(u, y) : y = f(u), u \in \mathbb{R}^m \}$$

of $f$.

2. [Murray, Li, Sastry] Consider the hopping robot in flight:

Here, $d$ is a fixed length, while $y$ is a variable. The total angular momentum is

$$I\dot{\theta} + M(d + y)^2(\dot{\theta} + \dot{\psi}).$$

Let us assume the initial total angular momentum is 0. By conservation of momentum, we have

$$I\dot{\theta} + M(d + y)^2(\dot{\theta} + \dot{\psi}) = 0$$

a nonholonomic constraint that can be phrased as

$$\begin{bmatrix} 0 & M(d + y)^2 & I + M(d + y)^2 \end{bmatrix} \begin{bmatrix} \dot{y} \\ \dot{\psi} \\ \dot{\theta} \end{bmatrix} = 0$$

This has the form $h(q)\dot{q} = 0$, where

$$q = \begin{bmatrix} y \\ \psi \\ \theta \end{bmatrix}.$$
By taking two vectors orthogonal to the row vector \( h(q) \), we get the state model

\[
\dot{q} = u_1 g_1(q) + u_2 g_2(q), \quad u_1 := \dot{y}, u_2 := \dot{\psi}
\]

\[
g_1(q) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad g_2(q) = \begin{bmatrix} 0 \\ \frac{1}{M(d+y)^2} \\ -\frac{M(d+y)^2}{I+M(d+y)^2} \end{bmatrix}.
\]

Study controllability of this system.
4.2 Brockett’s Theorem

Brockett’s theorem relates to the question of stabilizability of a nonlinear system. As we shall see, the unicycle cannot be stabilized by a continuous, memoryless state feedback.

Let’s recall the stability definitions for the system $\dot{x} = f(x)$:

1. A point $\bar{x}$ is an **equilibrium point** if $f(\bar{x}) = 0$. We can always shift $\bar{x}$ to the origin:

   \[
   \dot{x} = f(x), \quad \dot{\bar{x}} = f(\bar{x}) = f(\bar{x} + \bar{x}) =: \tilde{f}(\bar{x}).
   \]

   Hence we assume the origin is the equilibrium point.

   Let $B_\delta$ denote the open ball of radius $\delta$:

   \[
   B_\delta = \{ x : \| x \| < \delta \}.
   \]

2. The origin is **stable** if

   \[
   (\forall \varepsilon > 0)(\exists \delta > 0)(\forall x(0) \in B_\delta)(\forall t > 0)x(t) \in B_\varepsilon.
   \]

   The trajectory stays **arbitrarily** close to the origin $(\forall \varepsilon > 0)$ provided it starts **sufficiently** close to the origin $(\exists \delta > 0)$.

   ![Stability Diagram](image)

   If we define $\| x \|_\infty = \sup_{t \geq 0} \| x(t) \|$ and the space $L^\infty$ of all bounded signals, then stability is equivalent to continuity of the mapping

   \[
   x(0) \mapsto x(\cdot) : \mathbb{R}^n \longrightarrow L^\infty.
   \]

3. The origin is **asymptotically stable** if it is stable and

   \[
   (\exists \delta > 0)(\forall x(0) \in B_\delta) \lim_{t \to \infty} \| x(t) \| = 0.
   \]

Now we turn to the unicycle. Let’s first linearize it about the origin. Nonlinear model:

\[
\begin{align*}
\dot{x} &= v \cos(\theta) \\
\dot{y} &= v \sin(\theta) \\
\dot{\theta} &= \omega.
\end{align*}
\]
Linearized model:
\[
\begin{align*}
\dot{x} &= v \\
\dot{y} &= 0 \\
\dot{\theta} &= \omega.
\end{align*}
\]
Clearly, this is not stabilizable. So stabilizing the unicycle can’t be done merely by linearization about the equilibrium.

Quick almost-proof the unicycle can’t be stabilized by smooth state feedback

Let \( u = k(q) \), \( k \) smooth, \( k(0) = 0 \). Then the closed-loop system is
\[
\begin{align*}
\dot{x} &= k_1(x, y, \theta) \cos \theta \\
\dot{y} &= k_1(x, y, \theta) \sin \theta \\
\dot{\theta} &= k_2(x, y, \theta).
\end{align*}
\]
Then \( (x, y, \theta) \) is an equilibrium point iff
\[
\begin{align*}
0 &= k_1(x, y, \theta) \cos \theta \\
0 &= k_1(x, y, \theta) \sin \theta \\
0 &= k_2(x, y, \theta)
\end{align*}
\]
iff
\[
\begin{align*}
k_1(x, y, \theta) &= 0 \\
k_2(x, y, \theta) &= 0.
\end{align*}
\]
These are two smooth equations in 3 variables. If they define a one-dimensional manifold that passes through the origin, then the origin is not asymptotically stable, because if you start on this manifold, you don’t move. Further on this point, see the first exercise.

Next, can we see intuitively why the unicycle can’t be stabilized by smooth, time-invariant feedback control? Model:
\[
\begin{align*}
\dot{x} &= v \cos(\theta) \\
\dot{y} &= v \sin(\theta) \\
\dot{\theta} &= \omega \\
\Rightarrow \dot{q} &= f(q, u)
\end{align*}
\]
Assume \( u = k(q) \). Then
\[
\dot{q} = f(q, k(q)) =: F(q).
\]
Suppose
\[
(\exists \delta > 0)(\forall q(0) \in B_\delta) \lim_{t \to \infty} \|q(t)\| = 0,
\]
i.e., the origin is locally asymptotically attractive. Then the origin is not stable. To see this, imagine the motion starting here:

That is $x(0) = 0$, $y(0) = \delta$, $\theta(0) = 0$. It wouldn’t be true that

$$(\forall \varepsilon > 0)(\exists \delta > 0)x(0) = 0, y(0) = \delta, \theta(0) = 0 \implies (\forall t > 0)|\theta(t)| < \varepsilon.$$ 

Reason: $\theta(t)$ can’t be kept small: To go from

would require a large turn.

So it seems asymptotic stability is too strong a spec: Wouldn’t we be satisfied if the origin were just asymptotically attractive? Actually, this is tricky too: There is no continuous, time-invariant state feedback that makes the origin asymptotically attractive. I believe the intuitive reason is that it would require a discontinuous, or time-varying, control to handle the following initial condition,

since the controller would have to decide which way to turn.
Linearization about a Rectilinear Trajectory

Start with
\[
\begin{align*}
\dot{x} &= v \cos \theta \\
\dot{y} &= v \sin \theta \\
\dot{\theta} &= \omega.
\end{align*}
\]

Consider a nominal trajectory where the forward speed is a nonzero constant and the heading angle is constant:
\[
x_0(t), y_0(t), \theta(t) = \theta_0, v_0(t) = v_0 \neq 0, \omega_0(t) = 0.
\]

We have
\[
\begin{align*}
\dot{x}_0 &= v_0 \cos \theta_0 \\
\dot{y}_0 &= v_0 \sin \theta_0.
\end{align*}
\]

So
\[
\frac{dy_0}{dx_0} = \tan \theta_0,
\]

and hence
\[
y_0(t) = (\tan \theta_0)x_0(t) + c.
\]

That is, the motion is confined to a straight line in the \((x, y)\)-plane.

Fix such a solution and consider a variation about it:
\[
\begin{align*}
x(t) &= x_0(t) + \delta x(t) \\
y(t) &= y_0(t) + \delta y(t) \\
\theta(t) &= \theta_0 + \delta \theta(t) \\
v(t) &= v_0 + \delta v(t) \\
\omega(t) &= \delta \omega(t).
\end{align*}
\]

Then we have
\[
\begin{align*}
\dot{x}_0 + \dot{\delta x} &= (v_0 + \delta v) \cos(\theta_0 + \delta \theta) \\
\dot{y}_0 + \dot{\delta y} &= (v_0 + \delta v) \sin(\theta_0 + \delta \theta) \\
\dot{\theta}_0 + \dot{\delta \theta} &= \omega.
\end{align*}
\]

Expanding \(\cos, \sin\) and retaining only first-order variations gives
\[
\begin{align*}
\dot{\delta x} &= (\cos \theta_0) \delta v - (v_0 \sin \theta_0) \delta \theta \\
\dot{\delta y} &= (\sin \theta_0) \delta v + (v_0 \cos \theta_0) \delta \theta \\
\dot{\delta \theta} &= \omega.
\end{align*}
\]

This linear model has matrices
\[
A = \begin{bmatrix}
0 & 0 & -v_0 \sin \theta_0 \\
0 & 0 & v_0 \cos \theta_0 \\
0 & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
\cos \theta_0 & 0 \\
\sin \theta_0 & 0 \\
0 & 1
\end{bmatrix}
\]

and is therefore controllable for all \(v_0 \neq 0\) and all \(\theta_0\). By this means, the unicycle can easily be stabilized about the trajectory.
4.2.1 Statement of Brockett’s Theorem

The result is a necessary condition, which we’ll denote by BC (Brockett’s condition), for being able to stabilize

$$\dot{x} = f(x, u)$$

by a smooth, time-invariant state-feedback law $$u = k(x)$$. The unicycle doesn’t satisfy BC. Conclusion: The unicycle can’t be stabilized by such a control law.

Some terminology

Let $$\mathcal{X}$$ and $$\mathcal{Y}$$ be sets and let $$f : \mathcal{X} \rightarrow \mathcal{Y}$$ be a function. The **image** of $$f$$ is the set

$$\{f(x) : x \in \mathcal{X} \} = \{y \in \mathcal{Y} : (\exists x \in \mathcal{X}) y = f(x) \}.$$  

We say $$f$$ is **onto** (used as an adjective, also called **surjective**) if its image is all of $$\mathcal{Y}$$.

An $$m \times n$$ matrix $$A$$ is associated with a linear mapping $$L : \mathbb{R}^n \rightarrow \mathbb{R}^m$$ via $$L(x) = Ax$$. We frequently identify $$A$$ and $$L$$ and say $$A$$ is a linear mapping.

Linear Review

Consider

$$\dot{x} = Ax + Bu, \quad u = Fx$$

$$\dot{x} = (A + BF)x.$$ 

The origin, $$x = 0$$, is an equilibrium point. It is asymptotically stable iff

$$(\forall x(0)) \lim_{t \to \infty} x(t) = 0.$$ 

Equivalently, all the eigenvalues of $$A + BF$$ satisfy $$\text{Re } \lambda < 0$$. A necessary condition for asymptotic stability:

$$0$$ is not an eigenvalue of $$A + BF$$

$$\iff \quad \text{rank}(A + BF) = n$$

$$\iff \quad$$ the mapping $$A + BF$$ is onto.

Since

$$A + BF = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} I \\ F \end{bmatrix},$$

a necessary condition for stabilizability (existence of $$F$$) is that the mapping $$\begin{bmatrix} A & B \end{bmatrix}$$ is onto.

Back to the unicycle:

$$\begin{align*}
\dot{x} &= v \cos(\theta) \\
\dot{y} &= v \sin(\theta) \\
\dot{\theta} &= \omega.
\end{align*}$$
So

\[ f : (x, y, \theta, v, \omega) \mapsto (v \cos(\theta), v \sin(\theta), \omega). \]

The mapping \( f : \mathbb{R}^5 \rightarrow \mathbb{R}^3 \) does not satisfy the following condition:

**BC** The image of every open neighbourhood of the origin contains an open neighbourhood of the origin.

**Proof** It suffices to show that the mapping

\[
\begin{bmatrix}
\theta \\
v
\end{bmatrix} \mapsto \begin{bmatrix}
v \cos(\theta) \\
v \sin(\theta)
\end{bmatrix} : \mathbb{R}^2 \rightarrow \mathbb{R}^2
\]

does not satisfy BC. But this function maps the open set

\[ |\theta| < \pi/4, \ |v| < 1 \]

onto the “bowtie” set

which obviously doesn’t contain an open neighbourhood of the origin. \(\square\)

Intuitively, BC says that near the equilibrium \( x = 0, u = 0 \) there’s no restriction on the assignable vector field, that is, one can go in any direction. Notice that \( f : \mathbb{R}^5 \rightarrow \mathbb{R}^3 \) is surjective—that’s not the same as BC.

Now to Brockett’s Theorem. The plant is \( \dot{x} = f(x, u) \) with \( f(0, 0) = 0 \). The allowable control law is \( u = k(x) \) with \( k(0) = 0 \). Then the controlled system is \( \dot{x} = f(x, k(x)) =: F(x) \). If \( f \) and \( k \) are assumed only continuous, then existence and uniqueness of solutions are not immediate. So like Brockett, we assume \( f \) and \( k \) are smooth (continuously differentiable), though the theorem has been generalized since its original version.

**Theorem 21** (Brockett, 1983) A necessary condition for there to exist \( k \) so that the origin of \( \dot{x} = F(x) \) is (locally) asymptotically stable is that \( f \) satisfy BC.

**4.2.2 Proof**

Before the proof, let’s see the idea of the theorem in the very simple case \( \dim x = \dim u = 1 \). Assume the origin of \( \dot{x} = F(x) \) is (locally) asymptotically stable. Then for \( x \) sufficiently small, the motion must be towards the origin, i.e.,

\[ x > 0 \quad \Rightarrow \quad \dot{x} < 0 \]

\[ x < 0 \quad \Rightarrow \quad \dot{x} > 0, \]
that is,

\[ x > 0 \implies F(x) < 0 \]
\[ x < 0 \implies F(x) > 0. \]

Thus near the origin the graph of \( F \) must look like this:

![Graph of F near the origin](image)

From this picture it’s clear that, for every \( \varepsilon > 0 \), the image under \( F \) of the interval \((-\varepsilon, \varepsilon)\) contains an open interval centred at 0. Thus \( F \) satisfies BC. It remains to show that \( f \) does too.

We’ll prove this for general dimensions of \( x \) and \( u \).

**Lemma 19** Suppose \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \), \( k : \mathbb{R}^n \to \mathbb{R}^m \), \( f \) and \( k \) are continuous, \( f(0, 0) = 0 \), \( k(0) = 0 \), and \( F(x) = f(x, k(x)) \). If \( F \) satisfies BC, so does \( f \).

**Proof** (Orsi, Praly, Mareels) Assume \( F \) satisfies BC. Let \( \mathcal{N} \) be an open neighbourhood of 0 in \( \mathbb{R}^n \) and let \( \mathcal{M} \) be an open neighbourhood of 0 in \( \mathbb{R}^m \). We must show that \( f(\mathcal{N} \times \mathcal{M}) \) contains an open neighbourhood of 0 in \( \mathbb{R}^n \). Define \( \mathcal{N}_1 = k^{-1}(\mathcal{M}) \). Since \( k \) is continuous, \( \mathcal{N}_1 \) is an open set, and since \( k(0) = 0 \), \( \mathcal{N}_1 \) contains 0. Finally, define \( \mathcal{N}_2 = \mathcal{N} \cap \mathcal{N}_1 \), also an open neighbourhood of 0. Since \( F \) satisfies BC, \( F(\mathcal{N}_2) \) contains an open neighbourhood of 0, say \( \mathcal{V} \). Then

\[
\mathcal{V} \subset F(\mathcal{N}_2) \\
\subset f(\mathcal{N}_2 \times k(\mathcal{N}_2)) \text{ by def'n of } F \\
\subset f(\mathcal{N} \times k(\mathcal{N}_1)) \text{ since } \mathcal{N}_2 \subset \mathcal{N} \text{ and } \mathcal{N}_2 \subset \mathcal{N}_1 \\
\subset f(\mathcal{N} \times \mathcal{M}) \text{ since } k(\mathcal{N}_1) \subset \mathcal{M}.
\]

Thus \( f(\mathcal{N} \times \mathcal{M}) \) contains \( \mathcal{V} \). \( \square \)

Brockett’s proof is based on the concept of a fixed point of a function. He cites Lefschetz’s fixed-point theorem, but I think Brouwer’s fixed-point theorem, which is simpler, (almost) suffices. A 1-D example of Brouwer’s theorem is illustrated as follows:
Draw a unit square in the \((x, y)\)-plane as shown and draw the diagonal from \((0, 0)\) to \((1, 1)\). Draw any continuous curve from the left side of the square to the right side. Then the curve and the diagonal must intersect. So, if the curve is the graph of a function, \(f\), there must exist a point \(x\), called a fixed point of \(f\), satisfying \(f(x) = x\).

For a 2-D example, take two sheets of paper of the same size. Place one on a table. Crumple the other and place it on top of the first sheet. Brouwer’s theorem says that there must be at least one point on the top sheet that is directly above the corresponding point on the bottom sheet. A related experiment: Take a map of Toronto to any location in the city. Unfold the map and lay it out flat anywhere on the ground. Then there’s a (unique in this case) point on the map that lies directly above the corresponding point on the ground. [Ref: Josh]

For a 3-D example, Brouwer’s theorem says that if you take a cup of coffee and slosh it around, then after the coffee settles to rest there must be at least one coffee particle that is in exactly the same spot in the cup that it was before you did the sloshing.

**Theorem 22** (Brouwer, 1912) Let \(B\) be a nonempty, closed, bounded, convex subset of \(\mathbb{R}^n\) and let \(f\) be a continuous function \(\mathbb{R}^n \rightarrow \mathbb{R}^n\) such that \(f(B) \subset B\). Then there exists a point \(x\) in \(B\) such that \(f(x) = x\) (i.e., \(x\) is a fixed point of \(f\)).

Proof omitted. (It uses algebraic topology, in particular, homology.)

Let’s see how Brouwer’s theorem can be used in dynamical systems. First, a set \(B\) is said to be (positively) invariant with respect to a system \(\dot{x} = f(x)\) if \(x(0) \in B\) implies that \(x(t) \in B\) for every \(t > 0\).

**Lemma 20** Suppose \(B\) is a nonempty, closed, bounded, convex set in \(\mathbb{R}^n\) that is invariant with respect to the system \(\dot{x} = f(x)\), where \(f\) is smooth. Then this system has an equilibrium point in \(B\).

**Proof** Let \(\phi(t, x(0))\) denote the state \(x(t)\) at time \(t\) starting from the initial state \(x(0)\). Thus

\[
\phi(t, x(0)) = x(0) + \int_0^t f(\phi(\tau, x(0)))d\tau.
\]

For every \(t > 0\), \(\phi(t, \cdot)\) is a continuous function mapping \(B\) to \(B\); thus it has a fixed point, say \(q_t\), in \(B\), by Brouwer. From the preceding equation,

\[
0 = \int_0^t f(\phi(\tau, q_t))d\tau,
\]
and hence
\[ 0 = \frac{1}{\ell} \int_{0}^{\ell} f(\phi(\tau, q_t)) d\tau. \]

Take a sequence \( \{t_k\} \) of positive times converging to 0. Being bounded, the sequence \( \{q_{t_k}\} \) has a convergent subsequence, \( \{q_{t_{k_i}}\} \); let \( \bar{x} \) denote the limit of this subsequence. From the preceding equation we get
\[
0 = \lim_{i \to \infty} \frac{1}{t_{k_i}} \int_{0}^{t_{k_i}} f(\phi(\tau, q_{t_{k_i}})) d\tau = \lim_{i \to \infty} f(\phi(t_{k_i}, q_{t_{k_i}})) = \lim_{i \to \infty} f(q_{t_{k_i}}) = f(\bar{x}).
\]

Thus \( \bar{x} \) is an equilibrium point. \( \square \)

**Proof of Brockett’s theorem** (From Sontag, 1999) Assume the origin of \( \dot{x} = F(x) \) is asymptotically stable. Let \( \mathcal{N} \) be an open neighbourhood of the origin. We have to show that \( F(\mathcal{N}) \) contains an open neighbourhood of the origin; that is, for every \( w \) sufficiently small in norm, there exists \( x \in \mathcal{N} \) such that \( F(x) = w \).

Corresponding to \( \dot{x} = F(x) \) is a Lyapunov function, \( V(x) \). For \( c > 0 \) the set
\[
\{ x : V(x) = c \}
\]
is called a level set of \( V \) and
\[
\mathcal{B} = \{ x : V(x) \leq c \}
\]
is a sublevel set. Choose \( c \) so small that \( \mathcal{B} \subset \mathcal{N} \). Since \( V \) decreases along solutions of \( \dot{x} = F(x) \), the vector field \( F(x) \) must point into \( \mathcal{B} \) at all points on its boundary:

For every vector \( w \) sufficiently small in norm, the perturbed vector field \( F(x) - w \) too must point into \( \mathcal{B} \) at all points on its boundary. This means that \( \mathcal{B} \) is an invariant set for the system \( \dot{x} = F(x) - w \).

The sublevel set \( \mathcal{B} \) is definitely closed and bounded; let’s assume it’s convex, as in the picture, so that Brouwer’s theorem applies. (If it’s not convex, one needs Lefschetz’s theorem.) By Lemma 20, \( \dot{x} = F(x) - w \) has an equilibrium, \( \bar{x} \), in \( \mathcal{B} \), that is, \( F(\bar{x}) = w \). Done! \( \square \)

4.2.3 Example

Here’s an interesting example [Ref: Sontag] that shows that BC, though necessary, is not sufficient for existence of a continuous stabilizing controller. Consider the unicycle and suppose we want to go from a starting point to the origin via a circular path like this:
The circle is centred on the $y$-axis and is therefore uniquely defined by the starting point. The unicycle must be steered to be tangent to the circle at all times; the control input, $u$, is only the tangential speed. To get the equation of motion, consider an arbitrary point $(x, y)$, draw the circle, and then get the equation of the tangent to the circle at the point:

In this way we get

\[
\dot{x} = (x^2 - y^2)u \\
\dot{y} = 2xyu.
\]

Equivalently, in complex form using $z = x + jy$,

\[
\dot{z} = z^2u.
\]

This has the form $\dot{z} = f(z, u)$, where

\[ f : \mathbb{C} \times \mathbb{R} \rightarrow \mathbb{C}. \]

It remains to show that the origin in fact cannot be made asymptotically stable by a continuous state feedback.

**Proof** The system is

\[
\dot{x} = (x^2 - y^2)u \\
\dot{y} = 2xyu,
\]

or $\dot{q} = f(q)u$. For a proof by contradiction, suppose that $k$ is continuous and $u = k(q)$ makes the origin asymptotically stable. There is a neighbourhood of the origin throughout which the origin is asymptotically attractive. Since the $x$-axis is invariant, there must exist $\varepsilon > 0$ such that

\[ f(q)k(q) \text{ points left at } q = \left[ \begin{array}{c} \varepsilon \\ 0 \end{array} \right] \]
\[ f(q)k(q) \] points right at \( q = \begin{bmatrix} -\varepsilon \\ 0 \end{bmatrix}. \]

Since \( f(q) = \begin{bmatrix} \varepsilon^2 \\ 0 \end{bmatrix} \) at both these points, it must be that

\[ k(q) < 0 \text{ at } q = \begin{bmatrix} \varepsilon \\ 0 \end{bmatrix}, \quad k(q) > 0 \text{ at } q = \begin{bmatrix} -\varepsilon \\ 0 \end{bmatrix}. \]

Take any continuous path \( \gamma : [0, 1] \rightarrow \mathbb{R}^2 \) (in the nbhd) such that

\[ \gamma(0) = \begin{bmatrix} \varepsilon \\ 0 \end{bmatrix}, \quad \gamma(1) = \begin{bmatrix} -\varepsilon \\ 0 \end{bmatrix}, \quad \gamma(r) \neq 0, 0 < r < 1. \]

\[
\begin{array}{c}
\gamma \\
\downarrow \\
-\varepsilon \\
\varepsilon \\
\end{array}
\]

Defining \( \kappa = k \circ \gamma : [0, 1] \rightarrow \mathbb{R} \), we have

\[ \kappa(0) < 0, \quad \kappa(1) > 0. \]

Thus \( \kappa(\tau) = 0 \) for some \( \tau \in (0, 1) \), i.e.,

\[ k(\gamma(\tau)) = 0. \]

Define \( \overline{q} = \gamma(\tau) \neq 0 \). Then \( k(\overline{q}) = 0 \), so \( \overline{q} \) is another equilibrium near the origin. Contradiction. \( \square \)

### Exercises

1. This exercise\(^1\) is on linearization of the unicycle moving on a smooth curve. Start with the unicycle:

\[
\begin{align*}
\dot{x} &= v \cos \theta \\
\dot{y} &= v \sin \theta \\
\dot{\theta} &= \omega.
\end{align*}
\]

Also, consider a point \((x_0(t), y_0(t))\) that moves to trace out a smooth \((C^1)\) curve. Let \( r_0(t) \) denote the tangent vector, \( v_0(t) \) its magnitude (forward speed), and \( \theta_0(t) \) its angle:

\(^1\)Thanks to Manfredi Maggiore for this.
Thus

\[
\begin{align*}
\dot{x}_0 &= v_0 \cos \theta_0 \\
\dot{y}_0 &= v_0 \sin \theta_0.
\end{align*}
\]

If we define \( \omega_0 = \dot{\theta}_0 \), then we have

\[
\begin{align*}
\dot{x}_0 &= v_0 \cos \theta_0 \\
\dot{y}_0 &= v_0 \sin \theta_0 \\
\dot{\theta}_0 &= \omega_0,
\end{align*}
\]

which shows that the unicycle can move along the curve, perfectly tracking the point \((x_0(t), y_0(t))\). Now consider a differential in \( x \): \( x = x_0 + \delta x \). Likewise in \( y, \theta, v, \omega \). Neglecting higher order terms, obtain a linear, time-varying model with state \((\delta x, \delta y, \delta \theta)\) and input \((\delta v, \delta \omega)\). Is it controllable?

2. When do the two equations

\[
\begin{align*}
k_1(x, y, \theta) &= 0 \\
k_2(x, y, \theta) &= 0.
\end{align*}
\]

define a one-dimensional manifold?

3. If \( a < b \) are real numbers or \( \pm \infty \), the open interval \((a, b)\) is the set of all real numbers \( x \) satisfying \( a < x < b \). A subset \( \mathcal{V} \) of \( \mathbb{R} \) is an open set if for every \( x \) in \( \mathcal{V} \) there exists an open interval \( \mathcal{I} \) such that \( x \in \mathcal{I} \subseteq \mathcal{V} \). An open set containing a point \( x \) is called an open neighbourhood of \( x \).

Let \( f \) be a function \( \mathbb{R} \rightarrow \mathbb{R} \). If \( \mathcal{V} \) is a subset of \( \mathbb{R} \), the symbol \( f(\mathcal{V}) \) is called the image of \( \mathcal{V} \) under \( f \) and is defined as the set

\[
f(\mathcal{V}) = \{ f(x) : x \in \mathcal{V} \}.
\]

The function \( f \) is said to be an open function if \( f(\mathcal{V}) \) is an open set for every open set \( \mathcal{V} \), that is, the image under \( f \) of every open set is open, or \( f \) maps open sets to open sets. If \( \mathcal{V} \) is
a subset of $\mathbb{R}$, the symbol $f^{-1}(\mathcal{V})$ is called the pre-image of $\mathcal{V}$ under $f$ and is defined as the set 

$$f^{-1}(\mathcal{V}) = \{x : f(x) \in \mathcal{V}\}.$$ 

(There is no implication that $f$ is invertible.) The function $f$ is defined to be continuous if $f^{-1}(\mathcal{V})$ is an open set for every open set $\mathcal{V}$, that is, the pre-image under $f$ of every open set is open.

(a) Give an example of an $f$ that is continuous but not open.
(b) Give an example of an $f$ that is both continuous and open.
(c) Give an example of an $f$ that is neither continuous nor open.

4. (a) Let $T$ be a linear map from $\mathbb{R}^m$ to $\mathbb{R}^n$. Prove that the following conditions are equivalent:

i. $T$ is onto, that is, the image of $T$ is all of $\mathbb{R}^n$.
ii. $T$ satisfies BC.

(Recall that $T$ is onto iff every matrix representation has rank $n$.)

(b) Give an example of a nonlinear map $T$ where only one of the two conditions holds.

5. Start with the unicycle model

$$\dot{x} = v \cos(\theta)$$
$$\dot{y} = v \sin(\theta)$$
$$\dot{\theta} = \omega,$$

so

$$f : (x, y, \theta, v, \omega) \mapsto (v \cos(\theta), v \sin(\theta), \omega).$$

Transform the variables as follows:

$$z_1 = \theta$$
$$z_2 = x \cos(\theta) + y \sin(\theta)$$
$$z_3 = x \sin(\theta) - y \cos(\theta)$$
$$u_1 = \omega$$
$$u_2 = v - \omega z_3.$$

The corresponding vector field is

$$\tilde{f} : (z_1, z_2, z_3, u_1, u_2) \mapsto (u_1, u_2, z_2 u_1).$$

Show that $f : \mathbb{R}^5 \rightarrow \mathbb{R}^3$ is onto but $\tilde{f} : \mathbb{R}^5 \rightarrow \mathbb{R}^3$ is not. (Conclusion: Care must be taken in inferring properties of the first system when analyzing the second.)

6. Let $Q$ be a symmetric, positive definite matrix and $c$ a positive number. Prove that

$$B = \{x : x^T Q x \leq c\}$$

is convex.
7. (a) Prove that \( f : \mathbb{C} \times \mathbb{R} \rightarrow \mathbb{C}, (z,u) \mapsto z^2u \) satisfies BC.

(b) Show that the \( x \)-axis is an invariant subspace for any (real) \( u \). (Hint: \( \dot{z} = z^2u \).)

(c) The motion on the \( x \)-axis is \( \dot{x} = x^2u \). Is BC satisfied for this system? Can the origin be made asymptotically stable?

8. Show that the unicycle can’t be stabilized by smooth, time-invariant, dynamic control. More precisely, the setup is \( \dot{q} = f(q,u) \) and a time-invariant, dynamic controller looks like

\[
    u = k(w,q), \quad \dot{w} = h(w,q),
\]

that is, it’s a dynamic system with input \( q \), state \( w \), and output \( u \). Assume \( k \) and \( h \) are smooth.

9. Here’s a modified unicycle model:

\[
    \begin{align*}
    \dot{x} &= v \cos \theta \\
    \dot{y} &= v \sin \theta \\
    \dot{v} &= u_1 \\
    \dot{\theta} &= u_2.
    \end{align*}
\]

So the forward velocity is now a state variable and acceleration is an input.

(a) Let \( v(t), \theta(t) \) be constants, \( v_0 \neq 0, \theta_0 \), and find solutions \( x(t) = x_0(t), \ y(t) = y_0(t) \). Show that the motion is rectilinear in the \( (x,y) \)-plane.

(b) Linearize the model about the trajectory \( x_0(t), y_0(t), v_0, \theta_0 \): \( x(t) = x_0(t) + \delta x(t) \) etc.

(c) Show that the linearized model is controllable. Conclusion: This unicycle can be stabilized by linearization.

4.2.5 References


4.3 Stabilizing the Unicycle via Time-varying Control

As we saw, Brockett’s theorem tells us the unicycle can’t be (locally asymptotically) stabilized by a continuous time-invariant state-feedback controller. This suggests we try either discontinuous or time-varying. Here we’ll try the second.

The key point is, if you’re going to allow time-varying control, it’s better first to try periodically time-varying control. The reason is that LaSalle’s invariance principle (a very useful tool in Lyapunov stability analysis) applies to periodically time-varying systems but not time-varying in general, that is, systems of the form
\[ \dot{x} = f(t, x), \quad f(t, x) = f(t + T, x). \]

4.3.1 Transformation of Unicycle to a Simpler Form

The goal is to transform
\[
\begin{align*}
\dot{x} &= v \cos(\theta) \\
\dot{y} &= v \sin(\theta) \\
\dot{\theta} &= \omega
\end{align*}
\]
to
\[
\begin{align*}
\dot{z}_1 &= u_1 \\
\dot{z}_2 &= u_2 \\
\dot{z}_3 &= z_1 u_2.
\end{align*}
\]
In the transformation
\[
z_2 := \theta, \quad u_2 := \omega.
\]
For the rest of the derivation it’s convenient to simply notation:
\[
c := \cos(\theta), \quad s := \sin(\theta).
\]
Then \(c^2 + s^2 = 1\) and \(\dot{c} = -\omega s, \quad \dot{s} = \omega c = u_2 c\). Goal:
\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} = \begin{bmatrix} c & 0 \\ s & 0 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix} \quad \longrightarrow \quad \begin{bmatrix} \dot{z}_1 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & z_1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.
\]
Transformation:
\[
\begin{bmatrix}
z_1 \\
z_3
\end{bmatrix} = \begin{bmatrix} c & s \\ s & -c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & -z_3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}.
\]
Inverse transformation:
\[
\begin{bmatrix}
x \\
y
\end{bmatrix} = \begin{bmatrix} c & s \\ s & -c \end{bmatrix} \begin{bmatrix} z_1 \\ z_3 \end{bmatrix}, \quad \begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} 1 & z_3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.
\]
Proof it works

Differentiate wrt \( t \)

\[
\begin{bmatrix}
    z_1 \\
    z_3
\end{bmatrix} =
\begin{bmatrix}
    c & s \\
    s & -c
\end{bmatrix}
\begin{bmatrix}
    x \\
    y
\end{bmatrix}
\]

to get

\[
\begin{bmatrix}
    \dot{z}_1 \\
    \dot{z}_3
\end{bmatrix} = u_2 \begin{bmatrix}
    -s & c \\
    c & s
\end{bmatrix}
\begin{bmatrix}
    x \\
    y
\end{bmatrix} + \begin{bmatrix}
    c & s \\
    s & -c
\end{bmatrix}
\begin{bmatrix}
    \dot{x} \\
    \dot{y}
\end{bmatrix}.
\]

Substitute in

\[
\begin{bmatrix}
    x \\
    y
\end{bmatrix} = \begin{bmatrix}
    c & s \\
    s & -c
\end{bmatrix}
\begin{bmatrix}
    z_1 \\
    z_3
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
    \dot{x} \\
    \dot{y} \\
    \dot{\theta}
\end{bmatrix} = \begin{bmatrix}
    c & 0 \\
    s & 0
\end{bmatrix}
\begin{bmatrix}
    v \\
    \omega
\end{bmatrix} = \begin{bmatrix}
    c & 0 \\
    s & 0
\end{bmatrix}
\begin{bmatrix}
    1 & z_3 \\
    0 & 1
\end{bmatrix}
\begin{bmatrix}
    u_1 \\
    u_2
\end{bmatrix}
\]

and simplify:

\[
\begin{bmatrix}
    \dot{z}_1 \\
    \dot{z}_3
\end{bmatrix} = \begin{bmatrix}
    1 & 0 \\
    0 & 1
\end{bmatrix}
\begin{bmatrix}
    u_1 \\
    u_2
\end{bmatrix}.
\]

Complete equations

\[
\begin{bmatrix}
    \dot{x} \\
    \dot{y} \\
    \dot{\theta}
\end{bmatrix} = \begin{bmatrix}
    \cos(\theta) & 0 \\
    \sin(\theta) & 0 \\
    0 & 1
\end{bmatrix}
\begin{bmatrix}
    v \\
    \omega
\end{bmatrix} \quad \implies \quad
\begin{bmatrix}
    \dot{z}_1 \\
    \dot{z}_2 \\
    \dot{z}_3
\end{bmatrix} = \begin{bmatrix}
    1 & 0 \\
    0 & 1 \\
    0 & z_1
\end{bmatrix}
\begin{bmatrix}
    u_1 \\
    u_2
\end{bmatrix}.
\]

Transformation:

\[
\begin{bmatrix}
    z_1 \\
    z_2 \\
    z_3
\end{bmatrix} = \begin{bmatrix}
    \cos(\theta) & \sin(\theta) & 0 \\
    0 & 0 & 1 \\
    \sin(\theta) & -\cos(\theta) & 0
\end{bmatrix}
\begin{bmatrix}
    x \\
    y \\
    \theta
\end{bmatrix}
\]

\[
\begin{bmatrix}
    u_1 \\
    u_2
\end{bmatrix} = \begin{bmatrix}
    1 & -z_3 \\
    0 & 1
\end{bmatrix}
\begin{bmatrix}
    v \\
    \omega
\end{bmatrix}.
\]

Inverse transformation:

\[
\begin{bmatrix}
    x \\
    y \\
    \theta
\end{bmatrix} = \begin{bmatrix}
    \cos(\theta) & 0 & \sin(\theta) \\
    \sin(\theta) & 0 & -\cos(\theta) \\
    0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
    z_1 \\
    z_2 \\
    z_3
\end{bmatrix}
\]

\[
\begin{bmatrix}
    v \\
    \omega
\end{bmatrix} = \begin{bmatrix}
    1 & z_3 \\
    0 & 1
\end{bmatrix}
\begin{bmatrix}
    u_1 \\
    u_2
\end{bmatrix}.
\]

Note that the origin is preserved:

\[
\begin{bmatrix}
    x \\
    y \\
    \theta
\end{bmatrix} = 0 \iff \begin{bmatrix}
    z_1 \\
    z_2 \\
    z_3
\end{bmatrix} = 0.
\]
4.3.2 Time-varying Lyapunov Theory

Since we’re going to use time-varying control, we need to go over Lyapunov stability theory for time-varying systems. References:


The system model is \( \dot{x} = f(t, x) \). Let’s take the simplest case where \( f \) is defined for all \( t \in \mathbb{R} \) and \( x \in \mathbb{R}^n \). Let’s assume \( f \) is smooth enough that there exists a unique solution for every initial time \( t_0 \) and initial state \( x(t_0) \).

1. A point \( \bar{x} \) is an **equilibrium point** if \((\forall t)f(t, \bar{x}) = 0\). For the rest of the definitions we assume the origin is an equilibrium point.

2. The origin is **stable** if

\[
(\forall \varepsilon > 0)(\forall t_0) \quad (\exists \delta > 0) \quad (\forall x(t_0) \in B_\delta)(\forall t \geq t_0) x(t) \in B_\varepsilon.
\]

\( \delta \) depends on \( t_0 \)

3. The origin is **uniformly stable** if

\[
(\forall \varepsilon > 0) \quad (\exists \delta > 0) \quad (\forall t_0)(\forall x(t_0) \in B_\delta)(\forall t \geq t_0) x(t) \in B_\varepsilon.
\]

\( \delta \) doesn’t

4. The origin is **attractive** if

\[
(\forall t_0)(\exists \delta > 0)(\forall x(t_0) \in B_\delta) \lim_{t \to \infty} x(t) = 0.
\]

5. The origin is **uniformly attractive** if

\[
(\exists \delta > 0)(\forall t_0)(\forall x(t_0) \in B_\delta) \lim_{t \to \infty} x(t) = 0.
\]

6. The origin is **asymptotically stable** if it is stable and attractive.

7. The origin is **uniformly asymptotically stable** if it is uniformly stable and uniformly attractive.

Now we review the stability theorems in the time-invariant (autonomous) case of \( \dot{x} = f(x) \), again assuming the origin is an equilibrium point. Let \( V : \mathbb{R}^n \to \mathbb{R} \) be a function of class \( C^1 \). If \( x(t) \) is a solution of \( \dot{x} = f(x) \), then the time-derivative of \( V(x(t)) \) is

\[
\frac{d}{dt} V(x(t)) = \frac{\partial V}{\partial x}(x(t)) \dot{x}(t) = \frac{\partial V}{\partial x}(x(t)) f(x(t)).
\]

This motivates defining the function

\[
\dot{V} : \mathbb{R}^n \to \mathbb{R}, \quad \dot{V}(x) = \frac{\partial V}{\partial x}(x) f(x).
\]
Theorem 23 [Lyapunov, 1892] Suppose $\varepsilon > 0$ and $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is a function of class $C^1$ such that

\begin{align*}
V(0) &= 0 \\
(\forall x \in B_\varepsilon, x \neq 0)V(x) &> 0 \\
(\forall x \in B_\varepsilon)\dot{V}(x) &\leq 0.
\end{align*}

Then the origin is stable. Moreover, if

\begin{align*}
(\forall x \in B_\varepsilon, x \neq 0)\dot{V}(x) &< 0,
\end{align*}

then the origin is asymptotically stable.

It frequently happens that one can find a $V$ satisfying

\begin{align*}
V(0) &= 0 \\
(\forall x \in B_\varepsilon, x \neq 0)V(x) &> 0
\end{align*}

but only

\begin{align*}
(\forall x \in B_\varepsilon)\dot{V}(x) &\leq 0
\end{align*}

even though the origin is asymptotically stable. Then an additional condition is required to get the conclusion of asymptotic stability. This relates to LaSalle’s invariance principle. We shall need only a somewhat weaker result:

Theorem 24 [Barbashin and Krasovskii, 1950s] Suppose $\varepsilon > 0$ and $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is a function of class $C^1$ such that

\begin{align*}
V(0) &= 0 \\
(\forall x \in B_\varepsilon, x \neq 0)V(x) &> 0 \\
(\forall x \in B_\varepsilon)\dot{V}(x) &\leq 0.
\end{align*}

Suppose further that the only solution of $\dot{x} = f(x)$ that remains entirely in

$$
S = \{x : \dot{V}(x) = 0\}
$$

is the trivial solution. Then the origin is asymptotically stable.

Now we turn to the time-varying (non-autonomous) case of $\dot{x} = f(t, x)$, again assuming the origin is an equilibrium point. In general we need a time-varying Lyapunov function. Let $V : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a function of class $C^1$. If $x(t)$ is a solution of $\dot{x} = f(t, x)$, then the time-derivative of $V(t, x(t))$ is

\begin{align*}
\frac{d}{dt} V(t, x(t)) &= \frac{\partial V}{\partial t} (t, x(t)) + \frac{\partial V}{\partial x} (t, x(t)) \dot{x}(t) \\
&= \frac{\partial V}{\partial t} (t, x(t)) + \frac{\partial V}{\partial x} (t, x(t)) f(t, x(t)).
\end{align*}

This motivates defining the function

$$
\dot{V} : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}, \quad \dot{V}(t, x) = \frac{\partial V}{\partial t} (t, x) + \frac{\partial V}{\partial x} (t, x) f(t, x).
$$

A function $W : [0, \infty) \rightarrow [0, \infty)$ is said to be of class $K$ if it is continuous, strictly increasing, and $W(0) = 0$. 

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Theorem 25 [Lyapunov, 1892] Suppose $\varepsilon > 0$ and $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a function of class $C^1$ such that for some $W$ of class $K$ and all $t, x \in B_\varepsilon$

$$V(t, 0) = 0,$$

$$V(t, x) \geq W(\|x\|),$$

$$\dot{V}(t, x) \leq 0.$$

Then the origin is stable.

Theorem 26 [Persidski, 1933] Suppose $\varepsilon > 0$ and $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a function of class $C^1$ such that for some $W_1, W_2$ of class $K$ and all $t, x \in B_\varepsilon$

$$V(t, 0) = 0,$$

$$W_1(\|x\|) \geq V(t, x) \geq W_2(\|x\|),$$

$$\dot{V}(t, x) \leq 0.$$

Then the origin is uniformly stable.

Theorem 27 Suppose $\varepsilon > 0$ and $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a function of class $C^1$ such that for some $W_1, W_2, W_3$ of class $K$ and all $t, x \in B_\varepsilon$

$$V(t, 0) = 0,$$

$$W_1(\|x\|) \geq V(t, x) \geq W_2(\|x\|),$$

$$\dot{V}(t, x) \leq -W_3(\|x\|).$$

Then the origin is uniformly asymptotically stable.

Barbashin and Krasovskii’s theorem need not hold for time-varying systems:

Example Consider $\dot{x} = f(t, x) = -e^{-t}x$ where $x(t) \in \mathbb{R}$. The origin is uniformly stable: Take $V(t, x) = x^2/2$. Then

$$V(t, 0) = 0,$$

$$W_1(\|x\|) \geq V(t, x) \geq W_2(\|x\|),$$

where $W_1(x) = W_2(x) = x^2/2$, and

$$\dot{V}(t, x) = xf(t, x) = -e^{-t}x^2 \leq 0.$$

The only solution of $\dot{x} = f(t, x)$ that remains entirely in

$$S = \{(t, x) : \dot{V}(t, x) = 0\}$$
is the trivial solution. Yet the origin is \textit{not} asymptotically stable. To see this:

\[
\dot{x} = -e^{-t}x \implies \frac{dx}{x} = -e^{-t}dt \\
\implies \ln \frac{x(t)}{x(0)} = e^{-t} - 1 \\
\implies \ln \frac{x(\infty)}{x(0)} = -1 \\
\implies x(\infty) = e^{-1}x(0).
\]

Barbashin and Krasovskii’s theorem \textit{does} hold, however, for periodically time-varying systems:

\textbf{Theorem 28} [Krasovskii, 1959] \textit{Suppose} \(f(t, x)\) \textit{is periodic in} \(t\) \textit{of period} \(T\). \textit{Suppose} \(\varepsilon > 0\) \textit{and} \(V : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}\) \textit{is a function of class} \(C^1\), \textit{periodic of period} \(T\), \textit{such that for some} \(W_1, W_2\) \textit{of class} \(K\) \textit{and all} \(t, x \in B_\varepsilon\)

\[
V(t, 0) = 0, \\
W_1(\|x\|) \geq V(t, x) \geq W_2(\|x\|), \\
\dot{V}(t, x) \leq 0.
\]

\textit{Suppose further that the only solution of} \(\dot{x} = f(t, x)\) \textit{that remains entirely in}

\[S = \{x : \dot{V}(t, x) = 0\}\]

\textit{is the trivial solution. Then the origin is uniformly asymptotically stable.}

\textbf{4.3.3 Pomet’s Method of Stabilizing the Transformed Unicycle}

Our transformed unicycle is

\[
\begin{align*}
\dot{z}_1 &= u_1 \\
\dot{z}_2 &= u_2 \\
\dot{z}_3 &= z_1 u_2.
\end{align*}
\]

The familiar quadratic Lyapunov function is

\[
V(z) = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2 + \frac{1}{2}z_3^2.
\]

This doesn’t work here (try it), so out of the blue we propose the time-varying Lyapunov function

\[
V(t, z) = \frac{1}{2}[z_1 + (\cos t)z_3]^2 + \frac{1}{2}z_2^2 + \frac{1}{2}z_3^2.
\]

This is periodic in \(t\) and quadratic in \(z\). Clearly \(V(t, 0) = 0\).

\textbf{Claim} \textit{There exist} \(W_1, W_2\) \textit{of class} \(K\) \textit{such that for all} \(t, z\)

\[
W_1(\|z\|) \geq V(t, z) \geq W_2(\|z\|).
\]
Proof We have
\[
V(t, z) = \frac{1}{2} \begin{bmatrix} z_1 + (\cos t)z_3 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} z_1 + (\cos t)z_3 \\ z_2 \\ z_3 \end{bmatrix}
\]
so
\[
V(t, z) = \frac{1}{2} \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \cos t & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cos t \\ 0 & 1 & 0 \\ \cos t & 0 & 1 + \cos^2 t \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}
\]
Now for all \( t \)
\[
3I_3 \geq \begin{bmatrix} 1 & 0 & \cos t \\ 0 & 1 & 0 \\ \cos t & 0 & 1 + \cos^2 t \end{bmatrix} \geq \frac{1}{4} I_3.
\]
Thus
\[
\frac{3}{2} \|z\|^2 \geq V(t, z) \geq \frac{1}{8} \|z\|^2.
\]

Then
\[
\frac{d}{dt} V(t, z(t)) = [z_1 + (\cos t)z_3][\dot{z}_1 + (\cos t)\dot{z}_3 - (\sin t)z_3] + z_2 \dot{z}_2 + z_3 \dot{z}_3.
\]
Substituting in the state equations gives
\[
\frac{d}{dt} V(t, z(t)) = [z_1 + (\cos t)z_3][u_1 + (\cos t)z_1u_2 - (\sin t)z_3]
+ z_2u_2 + z_3z_1u_2
= [z_1 + (\cos t)z_3][u_1 - (\sin t)z_3]
+ \{z_2 + z_3z_1 + [z_1 + (\cos t)z_3](\cos t)z_1\}u_2.
\]
This suggests the control law
\[
u_1 - (\sin t)z_3 = -[z_1 + (\cos t)z_3]
\]
\[
u_2 = -\{z_2 + z_3z_1 + [z_1 + (\cos t)z_3](\cos t)z_1\}
\]
i.e.,
\[
u_1 = (\sin t)z_3 - z_1 - (\cos t)z_3
\]
\[
u_2 = -z_2 - z_3z_1 - [z_1 + (\cos t)z_3](\cos t)z_1
\]
for then
\[
\dot{V}(t, z) = -[z_1 + (\cos t)z_3]^2 - \{z_2 + z_3z_1 + [z_1 + (\cos t)z_3](\cos t)z_1\}^2 \leq 0.
\]
With these control laws, the controlled system is
\[
\begin{align*}
\dot{z}_1 & = (\sin t)z_3 - z_1 - (\cos t)z_3 \\
\dot{z}_2 & = -z_2 - z_3z_1 - [z_1 + (\cos t)z_3](\cos t)z_1 \\
\dot{z}_3 & = z_1\{-z_2 - z_3z_1 - [z_1 + (\cos t)z_3](\cos t)z_1\}
\end{align*}
\]
and the Lyapunov function is
\[
V(t, z) = \frac{1}{2}[z_1 + (\cos t)z_3]^2 + \frac{1}{2}z_2^2 + \frac{1}{2}z_3^2.
\]

**Claim** The only solution of these equations that remains entirely in 
\[ S = \{x : \dot{V}(t, x) = 0\} \]
is the trivial solution.

From Theorem 28 we conclude that the origin is uniformly asymptotically stable. Note that the control law has the form \( u = k(t, z) \), where \( k : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) and is smooth. In fact, \( k(t, z) \) is periodic in \( t \) and polynomial (multinomial) in \( z \).

### 4.3.4 Exercises
1. Show directly that the system
\[
\begin{align*}
\dot{z}_1 & = u_1 \\
\dot{z}_2 & = u_2 \\
\dot{z}_3 & = z_1u_2.
\end{align*}
\]
doesn’t satisfy Brockett’s condition.

2. Prove the last claim.

### 4.3.5 References


4.4 Stabilizing the Unicycle via Switching Control

As we saw, Brockett’s theorem tells us the unicycle can’t be (locally asymptotically) stabilized by a continuous time-invariant state-feedback controller. This suggests we try either discontinuous or time-varying. Here’ll we’ll try the first.

4.4.1 Transformation of Unicycle to Nonholonomic Integrator

The goal is to transform
\[
\begin{align*}
\dot{x} &= v \cos(\theta) \\
\dot{y} &= v \sin(\theta) \\
\dot{\theta} &= \omega
\end{align*}
\]
to
\[
\begin{align*}
\dot{z}_1 &= u_1 \\
\dot{z}_2 &= u_2 \\
\dot{z}_3 &= z_1 u_2 - z_2 u_1.
\end{align*}
\]
This latter model is called Brockett’s nonholonomic integrator. In the transformation
\[ z_2 := \theta, \quad u_2 := \omega. \]

For the rest of the derivation it’s convenient to simply notation:
\[ c := \cos(\theta), \quad s := \sin(\theta). \]

Then \( c^2 + s^2 = 1 \) and \( \dot{c} = -\omega s = -u_2 s, \quad \dot{s} = \omega c = u_2 c. \)

Goal:
\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} = \begin{bmatrix} c & 0 \\ s & 0 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix} \quad \longrightarrow \quad \begin{bmatrix}
\dot{z}_1 \\
\dot{z}_3
\end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -z_2 & z_1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.
\]

Transformation:
\[
\begin{bmatrix}
z_1 \\
z_3
\end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -z_2 & -2 \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & \frac{-1}{2}(z_1 z_2 + z_3) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}.
\]

Inverse transformation:
\[
\begin{bmatrix}
x \\
y
\end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} z_2 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} z_1 \\ z_3 \end{bmatrix}, \quad \begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2}(z_1 z_2 + z_3) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.
\]

Proof it works

Differentiate wrt \( t \)
\[
\begin{bmatrix}
z_1 \\
z_3
\end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -z_2 & -2 \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]

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to get
\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_3
\end{bmatrix}
= -u_2 \begin{bmatrix}
0 & 0 \\ 1 & 0
\end{bmatrix} \begin{bmatrix}
c & s \\ -s & c
\end{bmatrix} \begin{bmatrix}
x \\ y
\end{bmatrix}
+ u_2 \begin{bmatrix}
1 & 0 \\ -z_2 & -2
\end{bmatrix} \begin{bmatrix}
-s & c \\ -c & -s
\end{bmatrix} \begin{bmatrix}
x \\ y
\end{bmatrix}
+ \begin{bmatrix}
1 & 0 \\ -z_2 & -2
\end{bmatrix} \begin{bmatrix}
c & s \\ -s & c
\end{bmatrix} \begin{bmatrix}
\dot{x} \\ \dot{y}
\end{bmatrix}.
\]

Now substitute in
\[
\begin{bmatrix}
x \\ y
\end{bmatrix} = \begin{bmatrix}
c & -s \\ s & c
\end{bmatrix} \begin{bmatrix}
1 & 0 \\ -\frac{1}{2}z_2 & -\frac{1}{2}
\end{bmatrix} \begin{bmatrix}
z_1 \\ z_3
\end{bmatrix}
\]
\[
\begin{bmatrix}
\dot{x} \\ \dot{y}
\end{bmatrix} = \begin{bmatrix}
c & 0 \\ s & 0
\end{bmatrix} \begin{bmatrix}
v \\ \omega
\end{bmatrix} = \begin{bmatrix}
c & 0 \\ s & 0
\end{bmatrix} \begin{bmatrix}
1 & \frac{1}{2}(z_1z_2 + z_3) \\ 0 & 1
\end{bmatrix} \begin{bmatrix}
u_1 \\ u_2
\end{bmatrix}
\]
and simplify.

**Complete equations**
\[
\begin{bmatrix}
\dot{x} \\ \dot{y} \\ \dot{\theta}
\end{bmatrix} = \begin{bmatrix}
\cos(\theta) & 0 & v \\ \sin(\theta) & 0 & \omega \\ 0 & 1 & 0
\end{bmatrix} \begin{bmatrix}
u \\ \omega
\end{bmatrix} \quad \rightarrow \quad \begin{bmatrix}
\dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\ 0 & 1 & 1 \\ -z_2 & z_1 & 0
\end{bmatrix} \begin{bmatrix}
u_1 \\ u_2
\end{bmatrix}.
\]

Transformation:
\[
\begin{bmatrix}
z_1 \\ z_2 \\ z_3
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\ 0 & 1 & 0 \\ -\theta & 0 & -2
\end{bmatrix} \begin{bmatrix}
\cos(\theta) & \sin(\theta) & 0 \\ 0 & 0 & 1 \\ -\sin(\theta) & \cos(\theta) & 0
\end{bmatrix} \begin{bmatrix}
x \\ y \\ \theta
\end{bmatrix}
\]
\[
\begin{bmatrix}
u_1 \\ u_2
\end{bmatrix} = \begin{bmatrix}
1 & -\frac{1}{2}(z_1z_2 + z_3) \\ 0 & 1
\end{bmatrix} \begin{bmatrix}
v \\ \omega
\end{bmatrix}.
\]

Inverse transformation:
\[
\begin{bmatrix}
x \\ y \\ \theta
\end{bmatrix} = \begin{bmatrix}
\cos(\theta) & 0 & -\sin(\theta) \\ \sin(\theta) & 0 & \cos(\theta) \\ 0 & 1 & 0
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2}\theta & 0 & -\frac{1}{2}
\end{bmatrix} \begin{bmatrix}
z_1 \\ z_2 \\ z_3
\end{bmatrix}
\]
\[
\begin{bmatrix}
v \\ \omega
\end{bmatrix} = \begin{bmatrix}
1 & \frac{1}{2}(z_1z_2 + z_3) \\ 0 & 1
\end{bmatrix} \begin{bmatrix}
u_1 \\ u_2
\end{bmatrix}.
\]

Note that the origin is preserved:
\[
\begin{bmatrix}
x \\ y \\ \theta
\end{bmatrix} = 0 \iff \begin{bmatrix}
z_1 \\ z_2 \\ z_3
\end{bmatrix} = 0.
\]
4.4.2 Transformation of the Nonholonomic Integrator

Take polar coordinates in the \((z_1, z_2)\)-plane:

\[
\begin{align*}
  z_1 &= r \cos(\psi), \\
  z_2 &= r \sin(\psi)
\end{align*}
\]

Then

\[
\begin{bmatrix}
  z_1 \\
  z_2
\end{bmatrix} =
\begin{bmatrix}
  r \cos(\psi) \\
  r \sin(\psi)
\end{bmatrix}
\]

\[
\begin{bmatrix}
  \dot{z}_1 \\
  \dot{z}_2
\end{bmatrix} =
\begin{bmatrix}
  \cos(\psi) & -r \sin(\psi) \\
  \sin(\psi) & r \cos(\psi)
\end{bmatrix}
\begin{bmatrix}
  \dot{r} \\
  \dot{\psi}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  \cos(\psi) & -\sin(\psi) \\
  \sin(\psi) & \cos(\psi)
\end{bmatrix}
\begin{bmatrix}
  1 & 0 \\
  0 & r
\end{bmatrix}
\begin{bmatrix}
  \dot{r} \\
  \dot{\psi}
\end{bmatrix},
\]

and so

\[
\begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix} =
\begin{bmatrix}
  \cos(\psi) & -\sin(\psi) \\
  \sin(\psi) & \cos(\psi)
\end{bmatrix}
\begin{bmatrix}
  1 & 0 \\
  0 & r
\end{bmatrix}
\begin{bmatrix}
  \dot{r} \\
  \dot{\psi}
\end{bmatrix}.
\]

This suggests transforming the controls via

\[
\begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix} =
\begin{bmatrix}
  \cos(\psi) & -\sin(\psi) \\
  \sin(\psi) & \cos(\psi)
\end{bmatrix}
\begin{bmatrix}
  w_1 \\
  w_2
\end{bmatrix}
\]

so that

\[
\begin{bmatrix}
  1 & 0 \\
  0 & r
\end{bmatrix}
\begin{bmatrix}
  \dot{r} \\
  \dot{\psi}
\end{bmatrix} =
\begin{bmatrix}
  w_1 \\
  w_2
\end{bmatrix},
\]

i.e.,

\[
\begin{bmatrix}
  \dot{r} \\
  \dot{\psi}
\end{bmatrix} =
\begin{bmatrix}
  w_1 \\
  \frac{1}{r} w_2
\end{bmatrix}.
\]

Then

\[
\dot{z}_3 = z_1 u_2 - z_2 u_1 = r (u_2 \cos(\psi) - u_1 \sin(\psi)) = rw_2.
\]
Recap

\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2 \\
\dot{z}_3
\end{bmatrix} =
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
-z_2 & z_1
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix} \quad \rightarrow \quad
\begin{bmatrix}
\dot{r} \\
\dot{\psi} \\
\dot{z}_3
\end{bmatrix} =
\begin{bmatrix}
1 & 0 \\
0 & \frac{1}{r} \\
0 & r
\end{bmatrix}
\begin{bmatrix}
w_1 \\
w_2
\end{bmatrix}
\]

Inverse transformation:

\[
\begin{bmatrix}
z_1 \\
z_2 \\
z_3
\end{bmatrix} =
\begin{bmatrix}
r \cos(\psi) \\
r \sin(\psi) \\
z_3
\end{bmatrix}
\quad \begin{bmatrix}
u_1 \\
u_2
\end{bmatrix} =
\begin{bmatrix}
\cos(\psi) & -\sin(\psi) \\
\sin(\psi) & \cos(\psi)
\end{bmatrix}
\begin{bmatrix}
w_1 \\
w_2
\end{bmatrix}
\]

4.4.3 The Switching Controller

Since \(z_1^2 + z_2^2 = r^2\), it suffices to stabilize \((r, z_3)\); we can ignore \(\psi\). Problem reduces to stabilizing origin of

\[
\begin{align*}
\dot{r} &= w_1 \\
\dot{z}_3 &= rw_2.
\end{align*}
\]

Attempt #1

\(w_1 = -r, \quad w_2 = -z_3\)

Then

\[
\begin{align*}
\dot{r} &= -r \\
\dot{z}_3 &= -rz_3.
\end{align*}
\]

Clearly \(r(t) \rightarrow 0\) as \(t \rightarrow \infty\), \(\forall r(0)\). What about \(z_3\)? “Pole” of \(z_3\) system:

```
  X
```

“Pole” moves too quickly.

Example

\[
\begin{align*}
\dot{z}_3 &= -e^{-t}z_3 \\
\frac{dz_3}{z_3} &= -e^{-t}dt \\
\ln \left[ \frac{z_3(t)}{z_3(0)} \right] &= - \int_0^t e^{-\tau} d\tau \\
&= e^{-t} - 1 \\
z_3(t) &= e^{(e^{-t} - 1)}z_3(0) \\
&= e^{-1}z_3(0)
\end{align*}
\]
So origin is not asymptotically attractive. The fix: Make the “pole” go to zero more slowly.

**Remaining problem** For the system

\[
\begin{align*}
\dot{r} &= -r^2 \\
\dot{z}_3 &= -rz_3,
\end{align*}
\]

the \(z_3\)-axis \((r = 0)\) is an invariant subspace and all points in it are stationary. One solution: If \(r(0) = 0\), move away from the \(z_3\)-axis for some fixed time \(T > 0\), then apply the stabilizing controller:

This is a switching (discontinuous) controller. The value \(w_1 = 1\) is arbitrary; the choice \(w_1 = z_3(0)\) makes the origin asymptotically stable (Lyapunov sense) and not just asymptotically attractive.

**Summary: Discontinuous Stabilizer of Unicycle**

Transform unicycle

\[
\begin{align*}
\dot{x} &= v \cos(\theta) \\
\dot{y} &= v \sin(\theta) \\
\dot{\theta} &= \omega
\end{align*}
\]

to nonholonomic integrator

\[
\begin{align*}
\dot{z}_1 &= u_1 \\
\dot{z}_2 &= u_2 \\
\dot{z}_3 &= z_1 u_2 - z_2 u_1
\end{align*}
\]

and then to polar-coordinate system

\[
\begin{align*}
\dot{r} &= w_1 \\
\dot{z}_3 &= rw_2.
\end{align*}
\]

**Controller**

If \(r(0) \neq 0\), apply \(w_1 = -r^2, w_2 = -z_3\).

If \(r(0) = 0\), apply \(w_1 = z_3(0), w_2 = 0\) until \(t = T\), and thereafter \(w_1 = -r^2, w_2 = -z_3\).
4.4.4 Exercises

1. Show directly that the nonholonomic integrator

\[
\begin{align*}
\dot{z}_1 &= u_1 \\
\dot{z}_2 &= u_2 \\
\dot{z}_3 &= z_1 u_2 - z_2 u_1.
\end{align*}
\]

doesn’t satisfy Brockett’s condition.

2. Show that the system

\[
\begin{align*}
\dot{r} &= w_1 \\
\dot{z}_3 &= r w_2.
\end{align*}
\]

with the controller (Attempt #2)

\[
\begin{align*}
w_1 &= -r^2, \\
w_2 &= -z_3
\end{align*}
\]

has the property \((r(t), z_3(t)) \to (0, 0)\) as \(t \to \infty\) for all \(r(0) \neq 0\) and all \(z_3(0)\).

4.4.5 References

4.5 Stabilizing the Unicycle’s Position

For future use (the rendezvous problem for unicycles), here we look at an interesting scheme that stabilizes \((x, y)\) but not \(\theta\).

4.5.1 Setup

Consider a unicycle

\[
\begin{align*}
\dot{x} &= v \cos(\theta) \\
\dot{y} &= v \sin(\theta) \\
\dot{\theta} &= \omega
\end{align*}
\]

or in complex variable form

\[
\begin{align*}
\dot{z} &= ve^{j\theta} \\
\dot{\theta} &= \omega
\end{align*}
\]

or in Frenet-Serret form

\[
\begin{align*}
\dot{z} &= rv \\
\dot{r} &= s\omega \\
\dot{s} &= -r\omega.
\end{align*}
\]

Assume the unicycle can see a beacon at the origin. The position of the beacon is \(0 - z\). In the Frenet-Serret frame we have (subscript \(b\) for “beacon”)

\[
0 - z = x_br + y_bs,
\]

where

\[
\begin{align*}
x_b &= (0 - z)^T r = -z^T r \\
y_b &= (0 - z)^T s = -z^T s.
\end{align*}
\]

Thus we assume the unicycle can measure \(x_b, y_b\).
4.5.2 The Controller

The controller is defined as

\[ v(t) = kx_b(t) = -kz^T r, \quad \omega(t) = \cos(t), \]

where \( k \) is a small positive gain. Note that only the measurement \( x_b \) is needed, not \( y_b \).

The closed-loop equation is derived like this: Substitute \( v \) into \( \dot{z} = r \nu \) to get

\[ \dot{z} = -kz^T r \]

Define the \( 2 \times 2 \) matrix \( M = r r^T \). Then

\[ \dot{z} = -kM z. \]

Now look at \( M \):

\[ r = e^{i\theta} \]

\[ M = r r^T \]

\[ M = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix} \]

\[ = \begin{bmatrix} \cos^2(\theta) & \cos(\theta)\sin(\theta) \\ \cos(\theta)\sin(\theta) & \sin^2(\theta) \end{bmatrix} \]

Since \( \theta(t) = \theta(t_0) + \sin(t) \), so \( M(t) \) is a \( 2\pi \)-periodic function of \( t \).

Convergence of the unicycle to the origin reduces to studying a periodically time-varying linear system. This is a \textit{post facto} motivation for the control law \( \omega(t) = \cos(t) \).

Look at the function \( \cos^2(\theta(t)) \). Its average value over one period is

\[ \frac{1}{2\pi} \int_0^{2\pi} \cos^2(\theta(t))dt. \]

Likewise, the average of \( M(t) \) is

\[ \overline{M} := \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} \]

\[ = \begin{bmatrix} \frac{1}{2\pi} \int_0^{2\pi} \cos^2(\theta(t))dt & \frac{1}{2\pi} \int_0^{2\pi} \cos(\theta(t))\sin(\theta(t))dt \\ \frac{1}{2\pi} \int_0^{2\pi} \cos(\theta(t))\sin(\theta(t))dt & \frac{1}{2\pi} \int_0^{2\pi} \sin^2(\theta(t))dt \end{bmatrix}. \]

Claim 1 \( \overline{M} \) is positive definite.

\textbf{Proof} A symmetric matrix is positive definite iff its principle minors are positive. Since \( m_1 > 0 \), we just have to show \( \det(M) > 0 \), i.e., \( m_1 m_3 > m_2^2 \).

Define \( x(t) = \cos(\theta(t)), \ y(t) = \sin(\theta(t)) \) and the inner product

\[ \langle x, y \rangle = \frac{1}{2\pi} \int_0^{2\pi} x(t)y(t)dt. \]
Likewise for $\langle x, x \rangle$ and $\langle y, y \rangle$. Then $m_2^2 < m_1 m_3$ is equivalent to

$$\langle x, y \rangle^2 < \langle x, x \rangle \langle y, y \rangle.$$  

The inequality $\langle x, y \rangle^2 \leq \langle x, x \rangle \langle y, y \rangle$ is the Cauchy-Schwarz inequality and is always true. Equality holds iff $x$ is a scalar multiple of $y$ or vice versa. Neither is the case here—since $\theta(t)$ is time-varying, we can’t have $\cos(\theta(t)) = c \sin(\theta(t))$. $\Box$

With the periodically time-varying (PTV) linear system

$$\dot{z}(t) = -k M(t) z(t)$$

we associate the time-invariant (TI) linear system

$$\dot{z}(t) = -k \widetilde{M} z(t).$$

Convergence in the TI system is immediate since $\widetilde{M}$ is positive definite. We have to show that this implies convergence in the PTV system for small enough $k$. This uses averaging theory.

**The System** $\dot{x} = A(t)x$

We begin by reviewing the general linear time-varying system

$$\dot{x}(t) = A(t)x(t). \quad (4.1)$$

The transition matrix of (4.1) is the matrix that maps the state at one time, say $t_0$, to the state at another time, say $t$:

$$x(t) = \Phi(t,t_0) x(t_0).$$

In general, there’s no closed-form expression for $\Phi(t,t_0)$ in terms of $A(t)$ except in some special cases.

1. As you well know, if $A(t) = A$, a constant matrix, then

$$\Phi(t,t_0) = e^{A(t-t_0)}.$$

2. If $A(t)$ is a scalar ($1 \times 1$ matrix), then

$$\Phi(t,t_0) = e^{\int_{t_0}^t A(\tau)d\tau}.$$

3. If, for every value of $t_1$ and $t_2$, $A(t_2)$ and $\int_{t_1}^{t_2} A(\tau)d\tau$ commute, then

$$\Phi(t,t_0) = e^{\int_{t_0}^t A(\tau)d\tau}.$$

**Theorem 29** Let $A(t)$ be periodic of period $T$. Suppose that

$$\bar{A} = \frac{1}{T} \int_0^T A(\sigma)d\sigma$$

has all its eigenvalues in the half plane $\Re(s) < 0$. Then there exists $\varepsilon_0 > 0$ such that the origin of

$$\dot{x}(t) = \varepsilon A(t)x(t)$$

is exponentially stable for all $0 < \varepsilon < \varepsilon_0$.  

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Proof for $n = 1$ The case $n = 1$ is particularly easy. In that case, let’s write $a(t)$ for $A(t)$. The solution of $\dot{x}(t) = a(t)x(t)$ is

$$x(t) = e^{\int_{t_0}^{t} a(\tau) d\tau} x(t_0).$$

Each $t > t_0$ lies in one of the intervals

$$[t_0, t_0 + T), [t_0 + T, t_0 + 2T), \ldots$$

Say $t_0 + mT \leq t < t_0 + (m + 1)T$. Writing $\Delta t = t - (t_0 + mT)$, we have

$$x(t) = e^{\int_{t_0}^{t} a(\tau) d\tau} x(t_0) = e^{\int_{t_0}^{t_0+mT} a(\tau) d\tau + \int_{t_0+mT}^{t_0+mT+\Delta t} a(\tau) d\tau} x(t_0) = e^{mTa + \int_{t_0}^{t_0+mT} a(\tau) d\tau} x(t_0) = e^{mTa} \left( e^{\int_{t_0}^{t_0+mT} a(\tau) d\tau} x(t_0) \right).$$

Now $e^{\int_{t_0}^{t_0+mT} a(\tau) d\tau}$ is a bounded function for $0 \leq \Delta t < T$, and $e^{mTa}$ converges exponentially to 0 as $m \to \infty$, since $\tilde{a} < 0$. Thus $x(t)$ converges exponentially to 0 as $t \to \infty$.

Proof for general $n$: Omitted: See Khalil if you like. □
Chapter 5

Unicycle Formations

5.1 Up to Two Unicycles

In this section we look at some control laws for up to two unicycles.

5.1.1 One Unicycle and a Beacon

We now study a single unicycle together with a fixed beacon placed at the origin of the global frame. The goal is to get the unicycle to achieve a circular orbit around the beacon.

Again, the unicycle equations are

\[ \dot{z} = r, \]
\[ \dot{r} = s \omega, \]
\[ \dot{s} = -r \omega. \]

We study the control law

\[ \omega = -k_1 \left( \frac{z}{\|z\|} \cdot r \right) \left( \frac{z}{\|z\|} \cdot s \right) - k_2 \left( \frac{z}{\|z\|} \cdot s \right), \]

where the dot is the dot product and the gains \( k_1, k_2 \) are positive constants.

The rationale for this control law goes like this. First, suppose a counterclockwise circular orbit has been achieved:
Then $r \perp z$ and $(z/\|z\|) \cdot s = -1$, so

$$\omega = -k_1 \left( \frac{z}{\|z\|} \cdot r \right) \left( \frac{z}{\|z\|} \cdot s \right) - k_2 \left( \frac{z}{\|z\|} \cdot s \right) = k_2.$$ 

Thus we see that the function of $k_2$ is to keep the unicycle rotating counterclockwise to be tangent to the circle of orbit.

Secondly, let us look at the term

$$-k_1 \left( \frac{z}{\|z\|} \cdot r \right) \left( \frac{z}{\|z\|} \cdot s \right).$$

If the unicycle is moving outward from a circular orbit, the configuration is

and

$$\frac{z}{\|z\|} \cdot r > 0, \quad \frac{z}{\|z\|} \cdot s < 0$$

so

$$-k_1 \left( \frac{z}{\|z\|} \cdot r \right) \left( \frac{z}{\|z\|} \cdot s \right) > 0,$$

so the unicycle will want to rotate counterclockwise. Similar analysis if the unicycle is moving inward from a circular orbit.

Now we derive a useful equivalent form of the control law. For two 2-vectors $v, w$, their dot product can be expressed in terms of the complex numbers $v, w$ as follows:

$$v \cdot w = \text{Re}(v \bar{w}).$$

The control law is

$$\omega = -k_1 \left( \frac{z}{\|z\|} \cdot r \right) \left( \frac{z}{\|z\|} \cdot s \right) - k_2 \left( \frac{z}{\|z\|} \cdot s \right).$$

We have $r = e^{i\theta}$ and $s = je^{j\theta}$. It is convenient to write $z = \rho e^{i\psi}$, so the magnitude of $z$ is $\rho$ and the angle is $\psi + (\pi/2)$. Then

$$\frac{z}{\|z\|} \cdot r = \text{Re} \left( \frac{z}{\rho} r \right) = \text{Re} \left( je^{i\psi} e^{-j\theta} \right) = \sin(\theta - \psi)$$
and
\[
\frac{z}{\|z\|} \cdot s = \text{Re} \left( \frac{z}{\rho} \bar{s} \right) = \text{Re} \left( je^{j\psi}(-j)e^{-j\theta} \right) = \cos(\theta - \psi).
\]

Define \(\phi = \theta - \psi\). Then
\[
\omega = -k_1 \sin(\phi) \cos(\phi) - k_2 \cos(\phi).
\]

**Recap of key equations**

\[
\begin{align*}
\dot{z} &= e^{j\theta} \\
\dot{\theta} &= \omega \\
z &= \rho je^{j\psi} \\
\phi &= \theta - \psi \\
\omega &= -k_1 \sin(\phi) \cos(\phi) - k_2 \cos(\phi)
\end{align*}
\]

Thus
\[
\frac{d}{dt} \left( \rho je^{j\psi} \right) = e^{j\theta} \implies \dot{\rho} je^{j\psi} - \rho e^{j\psi} \dot{\psi} = e^{j\theta}
\]
\[
\implies \dot{\rho} - \rho \dot{\psi} = e^{j\phi}
\]
\[
\implies \dot{\psi} = -\frac{1}{\rho} \cos(\phi), \quad \dot{\rho} = \sin(\phi).
\]

And
\[
\dot{\phi} = \dot{\theta} - \dot{\psi} = \omega + \frac{1}{\rho} \cos(\phi)
\]
\[
= -k_1 \sin(\phi) \cos(\phi) - \left( k_2 - \frac{1}{\rho} \right) \cos(\phi).
\]

**Equations for stability analysis**

\[
\begin{align*}
\dot{\rho} &= \sin(\phi) \\
\dot{\phi} &= -k_1 \sin(\phi) \cos(\phi) - \left( k_2 - \frac{1}{\rho} \right) \cos(\phi)
\end{align*}
\]

Defined on \(\rho > 0\).

**Equilibrium points**

\[
\sin(\bar{\phi}) = 0, \quad -k_1 \sin(\bar{\phi}) \cos(\bar{\phi}) - \left( k_2 - \frac{1}{\rho} \right) \cos(\bar{\phi}) = 0
\]
\[
\Leftrightarrow
\]
\[
\bar{\phi} = 0, \pi; \quad \bar{\rho} = 1/k_2
\]
**Linearization at** $\phi = 0, \rho = 1/k_2$

\[ \begin{align*}
\dot{\rho} &= \sin(\phi) \\
\dot{\phi} &= -k_1 \sin(\phi) \cos(\phi) - (k_2 - \rho^{-1}) \cos(\phi)
\end{align*} \]

Jacobian:

\[
\begin{bmatrix}
0 & \cos(\phi) \\
-\rho^{-2} \cos(\phi) & -k_1 \left[ \cos^2(\phi) - \sin^2(\phi) \right] + (k_2 - \rho^{-1}) \sin(\phi)
\end{bmatrix}
\]

Jacobian at $\phi = 0, \rho = 1/k_2$:

\[
\begin{bmatrix}
0 & 1 \\
-k_2^2 & -k_1
\end{bmatrix}
\]

Thus, the nonlinear system is locally asymptotically stable $\forall k_1 > 0, k_2 > 0$.

**Linearization at** $\phi = \pi, \rho = 1/k_2$

The Jacobian at $\phi = 0, \rho = 1/k_2$ is

\[
\begin{bmatrix}
0 & -1 \\
k_2^2 & -k_1
\end{bmatrix}
\]

Thus, the nonlinear system is locally asymptotically stable $\forall k_1 > 0, k_2 > 0$.

Some remarks: Justh and Krishnaprasad actually propose the control law

\[
\omega = -k_1 \left( \frac{z}{\|z\|} \cdot r \right) \left( \frac{z}{\|z\|} \cdot s \right) - k_2(\|z\|) \left( \frac{z}{\|z\|} \cdot s \right),
\]

where $k_2$ is a nonlinear function of $\|z\|$, the distance to the beacon. (They allow $k_1$ to be a function of $\|z\|$ too, but there’s actually no advantage.) Then equilibrium points are characterized by

\[
\sin(\phi) = 0, \quad k_2(\rho) = 1/\rho.
\]

Assuming the latter equation has only isolated solutions in $\rho > 0$, and at least one solution there, and using the Lyapunov function

\[
V(\rho, \phi) = -\ln(\cos(\phi)) + \int [k_2(\rho) - \rho^{-1}] d\rho,
\]

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they prove under certain properties of \( k_1, k_2 \) that, for example, every trajectory starting in the region

\[
\rho > 0, \quad -\frac{\pi}{2} < \phi < \frac{\pi}{2}
\]

remains in that region and converges to an equilibrium point with \( \bar{\phi} = 0 \). Likewise, every trajectory starting in

\[
\rho > 0, \quad \frac{\pi}{2} < \phi < \frac{3\pi}{2}
\]

remains in that region and converges to an equilibrium point with \( \bar{\phi} = \pi \).

**Limiting motion of the unicycle**

Consider the case where \( \rho \) and \( \phi \) converge to \( \bar{\rho} > 0 \) and 0, respectively. What is the limiting motion of the unicycle? We have

\[
\dot{\psi} = -\frac{1}{\rho} \cos(\phi) \longrightarrow -\frac{1}{\bar{\rho}},
\]

so in the limit \( z(t) \) rotates clockwise at constant speed; the unicycle revolves clockwise around the beacon at constant speed. Since \( \phi = \theta - \psi \), in the limit \( \theta = \psi \) and

\[
\angle z = \angle r + \frac{\pi}{2},
\]

so in the limit the unicycle rotates clockwise at constant speed:

Similarly, when \( \rho \) and \( \phi \) converge to \( \bar{\rho} > 0 \) and \( \pi \), respectively, the limiting motion is
5.1.2 Two Unicycles, Rectilinear Motion

We consider two unicycles with the goal for them to move eventually in the same direction. This is a kind of agreement problem: Agree upon a common heading. The two models are

\[
\begin{align*}
\dot{z}_1 &= r_1 \\
\dot{r}_1 &= s_1 \omega_1 \\
\dot{s}_1 &= -r_1 \omega_1 \\
\end{align*}
\]

and

\[
\begin{align*}
\dot{z}_2 &= r_2 \\
\dot{r}_2 &= s_2 \omega_2 \\
\dot{s}_2 &= -r_2 \omega_2. \\
\end{align*}
\]

We study the control laws

\[
\begin{align*}
\omega_1 &= -k_1 \left( \frac{z_1 - z_2}{\|z_1 - z_2\|} \cdot r_1 \right) \left( \frac{z_1 - z_2}{\|z_1 - z_2\|} \cdot s_1 \right) \\
&\quad - k_2(\|z_1 - z_2\|) \left( \frac{z_1 - z_2}{\|z_1 - z_2\|} \cdot s_1 \right) + k_3 r_2 \cdot s_1 \\
\omega_2 &= -k_1 \left( \frac{z_2 - z_1}{\|z_2 - z_1\|} \cdot r_2 \right) \left( \frac{z_2 - z_1}{\|z_2 - z_1\|} \cdot s_2 \right) \\
&\quad - k_2(\|z_2 - z_1\|) \left( \frac{z_2 - z_1}{\|z_2 - z_1\|} \cdot s_2 \right) + k_3 r_1 \cdot s_2. \\
\end{align*}
\]

Or, with \( z = z_2 - z_1 \),

\[
\begin{align*}
\omega_1 &= -k_1 \left( -\frac{z}{\|z\|} \cdot r_1 \right) \left( -\frac{z}{\|z\|} \cdot s_1 \right) - k_2(\|z\|) \left( -\frac{z}{\|z\|} \cdot s_1 \right) + k_3 r_2 \cdot s_1 \\
\omega_2 &= -k_1 \left( \frac{z}{\|z\|} \cdot r_2 \right) \left( \frac{z}{\|z\|} \cdot s_2 \right) - k_2(\|z\|) \left( \frac{z}{\|z\|} \cdot s_2 \right) + k_3 r_1 \cdot s_2. \\
\end{align*}
\]

Equivalent form of control law

Write

\[
\begin{align*}
z &= \rho j e^{i \psi} \\
r_1 &= e^{i \theta_1} \\
r_2 &= e^{i \theta_2} \\
\phi_1 &= \theta_1 - \psi \\
\phi_2 &= \theta_2 - \psi. \\
\end{align*}
\]

Then

\[
\begin{align*}
\omega_1 &= -k_1 \left( -\frac{z}{\|z\|} \cdot r_1 \right) \left( -\frac{z}{\|z\|} \cdot s_1 \right) - k_2(\|z\|) \left( -\frac{z}{\|z\|} \cdot s_1 \right) + k_3 r_2 \cdot s_1 \\
&= -k_1 \sin(\phi_1) \cos(\phi_1) + k_2(\rho) \cos(\phi_1) + k_3 \sin(\theta_2 - \theta_1) \\
\omega_2 &= -k_1 \left( \frac{z}{\|z\|} \cdot r_2 \right) \left( \frac{z}{\|z\|} \cdot s_2 \right) - k_2(\|z\|) \left( \frac{z}{\|z\|} \cdot s_2 \right) + k_3 r_1 \cdot s_2 \\
&= -k_1 \sin(\phi_2) \cos(\phi_2) - k_2(\rho) \cos(\phi_2) + k_3 \sin(\theta_1 - \theta_2). \\
\end{align*}
\]
Recap of key equations

\[
\begin{align*}
\dot{z} &= e^{j\theta_2} - e^{j\theta_1} \\
\dot{\theta}_1 &= \omega_1 \\
\dot{\theta}_2 &= \omega_2 \\
z &= \rho je^{j\psi} \\
\phi_1 &= \theta_1 - \psi \\
\phi_2 &= \theta_2 - \psi \\
\omega_1 &= -k_1 \sin(\phi_1) \cos(\phi_1) + k_2(\rho) \cos(\phi_1) + k_3 \sin(\theta_2 - \theta_1) \\
\omega_2 &= -k_1 \sin(\phi_2) \cos(\phi_2) - k_2(\rho) \cos(\phi_2) + k_3 \sin(\theta_1 - \theta_2)
\end{align*}
\]

Thus in turn

\[
\begin{align*}
\frac{d}{dt} (\rho je^{j\psi}) &= e^{j\theta_2} - e^{j\theta_1} \\
\dot{\rho} je^{j\psi} - \rho e^{j\psi} \dot{\psi} &= e^{j\theta_2} - e^{j\theta_1} \\
\dot{j} \rho - \rho \dot{\psi} &= \cos(\phi_2) + j \sin(\phi_2) - \cos(\phi_1) - j \sin(\phi_1) \\
\dot{\psi} &= -\frac{1}{\rho} [\cos(\phi_2) - \cos(\phi_1)], \quad \dot{\rho} = \sin(\phi_2) - \sin(\phi_1).
\end{align*}
\]

And, since \(\theta_2 - \theta_1 = \phi_2 - \phi_1\),

\[
\begin{align*}
\dot{\phi}_1 &= \dot{\theta}_1 - \dot{\psi} \\
&= \omega_1 + \frac{1}{\rho} [\cos(\phi_2) - \cos(\phi_1)] \\
&= -k_1 \sin(\phi_1) \cos(\phi_1) + k_2(\rho) \cos(\phi_1) \\
&\quad + k_3 \sin(\phi_2 - \phi_1) + \frac{1}{\rho} [\cos(\phi_2) - \cos(\phi_1)] \\
\dot{\phi}_2 &= \dot{\theta}_2 - \dot{\psi} \\
&= \omega_2 + \frac{1}{\rho} [\cos(\phi_2) - \cos(\phi_1)] \\
&= -k_1 \sin(\phi_2) \cos(\phi_2) + k_2(\rho) \cos(\phi_2) \\
&\quad + k_3 \sin(\phi_1 - \phi_2) + \frac{1}{\rho} [\cos(\phi_2) - \cos(\phi_1)].
\end{align*}
\]

Equations for stability analysis

\[
\begin{align*}
\dot{\rho} &= \sin(\phi_2) - \sin(\phi_1) \\
\dot{\phi}_1 &= -k_1 \sin(\phi_1) \cos(\phi_1) + k_2(\rho) \cos(\phi_1) \\
&\quad + k_3 \sin(\phi_2 - \phi_1) + \frac{1}{\rho} [\cos(\phi_2) - \cos(\phi_1)] \\
\dot{\phi}_2 &= -k_1 \sin(\phi_2) \cos(\phi_2) + k_2(\rho) \cos(\phi_2) \\
&\quad + k_3 \sin(\phi_1 - \phi_2) + \frac{1}{\rho} [\cos(\phi_2) - \cos(\phi_1)].
\end{align*}
\]
Defined on $\rho > 0$.

**Equilibrium points**
Assume there exists a unique $\rho > 0$ such that $k_2(\rho) = 0$. Then

$$(\rho, \phi_1, \phi_2) = (\rho, 0, 0)$$

is an equilibrium point. There are others (a continuum), namely,

$$(\rho, \phi_1, \phi_2) = \left(\frac{\pi}{2}, \frac{\pi}{2}, \rho > 0\right)$$

and

$$(\rho, \phi_1, \phi_2) = \left(\frac{\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \rho > 0\right).$$

**Linearization at** $(\rho, \phi_1, \phi_2) = (\rho, 0, 0)$

$$\begin{align*}
\dot{\rho} &= \sin(\phi_2) - \sin(\phi_1) \\
\dot{\phi}_1 &= -k_1 \sin(\phi_1) \cos(\phi_1) + k_2(\rho) \cos(\phi_1) \\
&\quad + k_3 \sin(\phi_2 - \phi_1) + \frac{1}{\rho} \cos(\phi_2 - \cos(\phi_1)) \\
\dot{\phi}_2 &= -k_1 \sin(\phi_2) \cos(\phi_2) + k_2(\rho) \cos(\phi_2) \\
&\quad + k_3 \sin(\phi_1 - \phi_2) + \frac{1}{\rho} \cos(\phi_2 - \cos(\phi_1))
\end{align*}$$

Jacobian is

$$\begin{bmatrix}
0 & -1 & 1 \\
k_1' \frac{\rho}{\rho} & -(k_1 + k_3) & k_3 \\
-k_2' \frac{\rho}{\rho} & k_3 & -(k_1 + k_3)
\end{bmatrix}.$$

By Routh-Hurwitz, nonlinear system is locally asymptotically stable $\forall k_1 > 0, k_2' > 0, k_3 > 0$.

**Limiting motion of the two unicycles**
Consider the case where $(\rho, \phi_1, \phi_2)$ converge to $(\rho, 0, 0)$. What is the limiting motion of the unicycles? We have

$$\dot{\psi} = -\frac{1}{\rho} \cos(\phi_2) - \cos(\phi_1) \to 0,$$

so in the limit $\psi$ = constant and $z(t)$ has constant direction. Since $\phi_1 = \theta_1 - \psi$ and $\phi_2 = \theta_2 - \psi$, in the limit $\theta_1 = \theta_2 = \psi$ and so $\angle r_1 = \angle r_2$. The limiting motion is rectilinear: The unicycles move beside each other, distance $\rho$ apart, in the same, constant direction:
5.1.3 Formation Equilibria

We now look at the equilibrium formations that are possible for general steering laws. We’ll see that there are only two: Rectilinear motion, where all unicycles are moving in the same constant direction; circular motion, where all unicycles are moving in the same direction on the same circle. We emphasize that this is a consequence of the assumption that all unicycles have unit forward speed.

We’ll do only the case of two unicycles. So the possible equilibrium formations are as follows:

The vector notation is as follows:

\[
\begin{align*}
\mathbf{r}_2(t) &= c_1 \mathbf{r}_1(t) + c_2 \mathbf{s}_1(t) \\
\mathbf{z}_2(t) - \mathbf{z}_1(t) &= c_3 \mathbf{r}_1(t) + c_4 \mathbf{s}_1(t).
\end{align*}
\]

It’s much simpler to use complex numbers instead of vectors. Then the vector equations can be
replaced by

\begin{align*}
    r_2(t) &= c_1 r_1(t) + c_2 s_1(t) \\
    z_2(t) - z_1(t) &= c_3 r_1(t) + c_4 s_1(t).
\end{align*}

Recalling that \( s_1 = j r_1 \), we have the existence of (complex) \( c_5, c_6 \) such that

\begin{align*}
    r_2(t) &= c_5 r_1(t) \\
    z_2(t) - z_1(t) &= c_6 r_1(t).
\end{align*}

Claim \( \omega_1 = \omega_2 = \text{constant} \)

Proof We have

\begin{align*}
    r_2 &= c_5 r_1 \implies \dot{r}_2 = c_5 \dot{r}_1 \\
    &\implies \omega_2 s_2 = c_5 \omega_1 s_1 \\
    &\implies j \omega_2 r_2 = j c_5 \omega_1 r_1 \\
    &\implies j \omega_2 c_5 r_1 = j c_5 \omega_1 r_1 \\
    &\implies \omega_2 = \omega_1.
\end{align*}

And

\begin{align*}
    z_2 - z_1 &= c_6 r_1 \implies \dot{z}_2 - \dot{z}_1 = c_6 \dot{r}_1 \\
    &\implies r_2 - r_1 = c_6 \omega_1 s_1 \\
    &\implies r_2 - r_1 = j c_6 \omega_1 r_1 \\
    &\implies c_5 r_1 - r_1 = j c_6 \omega_1 r_1 \\
    &\implies (c_5 - 1 - j c_6 \omega_1) r_1 = 0 \\
    &\implies c_5 - 1 - j c_6 \omega_1 = 0 \\
    &\implies \omega_1 = \text{constant}.
\end{align*}

Claim If \( \omega_1 = \omega_2 = 0 \), the motion is rectilinear.

Proof Clearly the unicycles move in straight lines. The lines are parallel because

\( z_2 - z_1 = c_6 r_1 \implies |z_2 - z_1| = |c_6| = \text{constant} \).

Claim If \( \omega_1 = \omega_2 \neq 0 \), the motion is circular.

Proof Let \( \omega = \omega_1 = \omega_2 \). We have

\begin{align*}
    \dot{\theta}_1 = \omega, \quad \dot{z}_1 &= e^{i \theta_1} \\
    \implies \theta_1(t) &= \theta_1(0) + \omega t, \quad \dot{z}_1 = e^{i \theta_1(0)} e^{i \omega t} \\
    \implies (\exists c_7) z_1(t) &= c_7 + \frac{1}{j \omega} e^{i \theta_1(0)} e^{i \omega t}.
\end{align*}
So \( z_1(t) \) moves on the circle centre \( c_7 \), radius \( 1/|\omega| \), counterclockwise if \( \omega > 0 \). Likewise, 
\[
(\exists \epsilon_8) z_2(t) = c_8 + \frac{1}{j\omega} e^{j\epsilon_2(0)} e^{j\omega t},
\]
so \( z_2(t) \) moves on the circle centre \( c_8 \), radius \( 1/|\omega| \), also counterclockwise if \( \omega > 0 \). It remains to show that \( c_7 = c_8 \).
\[
z_2 - z_1 = c_6 r_1 \implies c_8 + \frac{1}{j\omega} c_2(t) - c_7 - \frac{1}{j\omega} c_1(t) = c_6 r_1(t) \]
\[
\implies c_8 + \frac{1}{j\omega} c_2(t) - c_7 - \frac{1}{j\omega} r_1(t) = c_6 r_1(t) \]
\[
\implies c_8 + \frac{1}{j\omega} c_5 r_1(t) - c_7 - \frac{1}{j\omega} r_1(t) = c_6 r_1(t) \]
\[
\implies c_8 - c_7 = \left( -\frac{1}{j\omega} c_5 + \frac{1}{j\omega} + c_6 \right) r_1(t) \]
\[
\implies c_8 - c_7 = 0.
\]
\[ \Box \]

5.1.4 Conclusions

Each unicycle must sense all others, i.e., the sensor graph is full of links. Global stability analysis only for \( n = 2 \). Extension:


What about constant but different forward speeds?

5.1.5 Exercises

1. An orthogonal matrix acts as a rotation. The other relevant motion in rigid-body mechanics is translation. If \( R \) is a \( 2 \times 2 \) orthogonal matrix and \( b \) a vector in the plane, the mapping
\[
r \mapsto s = Rr + b
\]
performs rotation + translation. The set of all such mappings forms a Lie group, called the special Euclidean group and denoted \( SE(2, \mathbb{R}) \). Another way to write the operation is
\[
\begin{bmatrix} s \\ 1 \end{bmatrix} = \begin{bmatrix} R & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r \\ 1 \end{bmatrix}.
\]
The \( 3 \times 3 \) matrix \( \begin{bmatrix} R & b \\ 0 & 1 \end{bmatrix} \) is invertible, and in this way \( SE(2, \mathbb{R}) \) is a subgroup of the group of \( 3 \times 3 \) invertible matrices.

Corresponding to the Frenet-Serret equations
\[
\begin{align*}
\dot{z} &= r \\
\dot{r} &= s_\omega \\
\dot{s} &= -r_\omega,
\end{align*}
\]

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it is natural to define the $3 \times 3$ matrix

$$g = \begin{bmatrix} r & s & z \\ 0 & 0 & 1 \end{bmatrix}.$$  

Note that $\begin{bmatrix} r & s \end{bmatrix}$ is an orthogonal matrix.

(a) Look up the definition of a Lie group.
(b) Write the Frenet-Serret equations as a single equation of the form $\dot{g} = g \xi$.
(c) The matrix $\xi$ belongs to the Lie algebra associated with the Lie group $SE(2, \mathbb{R})$. Look up the definition of the Lie algebra of a Lie group. What is the Lie algebra of $SE(2, \mathbb{R})$?

2. Consider the unicycle in the Frenet-Serret form

$$\begin{align*}
\dot{z} &= rv \\
\dot{r} &= s\omega \\
\dot{s} &= -r\omega.
\end{align*}$$

Define a new position vector, $p = z + r$:

![Diagram showing the unicycle with vectors](image)

Then

$$\dot{p} = rv + s\omega.$$  

Let’s define the right-hand side to be a new control input:

$$u := rv + s\omega.$$  

The transformed system is a simple linear integrator $\dot{p} = u$, and the control law $u = -p$ obviously drives $p$ to the origin from any starting point.

(a) Derive formulas for $v$ and $\omega$ in terms of $z, r, s$.
(b) Study the unicycle controlled in this way. In particular, what can you say about $\lim_{t \to \infty} z(t)$?
3. Study the two unicycles with the control laws

\[
\omega_1 = -k_1 \left( \frac{z_1 - z_2}{\|z_1 - z_2\|} \cdot r_1 \right) - k_2(\|z_1 - z_2\|) \left( \frac{z_1 - z_2}{\|z_1 - z_2\|} \cdot s_1 \right)
\]

\[
\omega_2 = -k_1 \left( \frac{z_2 - z_1}{\|z_2 - z_1\|} \cdot r_2 \right) - k_2(\|z_2 - z_1\|) \left( \frac{z_2 - z_1}{\|z_2 - z_1\|} \cdot s_2 \right).
\]

Show there’s a locally stable circular limiting formation for certain gains.

5.1.6 References


5.2 Pseudo-linearization and Cyclic Pursuit

A brief discussion of the following idea: Linearize the unicycles about points just ahead; then do cyclic pursuit.

Consider \( n > 1 \) unicycles:

\[
\begin{align*}
\dot{x}_i &= v_i \cos(\theta_i) \\
\dot{y}_i &= v_i \sin(\theta_i) \\
\dot{\theta}_i &= \omega_i
\end{align*}
\]

or

\[
\begin{align*}
\dot{z}_i &= r_i v_i \\
\dot{\theta}_i &= \omega_i
\end{align*}
\]

or

\[
\begin{align*}
\dot{z}_i &= r_i v_i \\
\dot{r}_i &= s_i \omega_i \\
\dot{s}_i &= -r_i \omega_i
\end{align*}
\]

Unicycle \( i \) can measure only the relative positions of sensed vehicles with respect to its own Frenet-Serret frame. Suppose unicycle \( i \) can sense unicycle \( m \). Write the relative position \( z_m - z_i \) in \( i \)'s frame:

\[
z_m - z_i = x_{im} r_i + y_{im} s_i.
\]

Thus we assume unicycle \( i \) can measure only \( x_{im}, y_{im} \) about unicycle \( m \). Note that

\[
\begin{align*}
x_{im} &= (z_m - z_i)^T r_i \\
y_{im} &= (z_m - z_i)^T s_i.
\end{align*}
\]

Again \( N_i \) denotes those vehicles sensed by unicycle \( i \). An allowable controller looks like

\[
v_i = g_i \left( t, \{x_{im}, y_{im}\} \subset N_i \right), \quad \omega_i = h_i \left( t, \{x_{im}, y_{im}\} \subset N_i \right),
\]

where \( g_i, h_i \) are smooth functions of their arguments, and \( g_i \) is such that

\[
\{N_i = \phi\} \implies \{v_i = 0\},
\]

i.e., there is zero forward velocity when the unicycle cannot sense any other vehicle, and

\[
\{(\forall m \in N_i) z_m = z_i\} \implies \{v_i = 0\},
\]

i.e., there is zero forward velocity when the unicycle’s neighbours have all converged to its position.
Back to one unicycle:

\[
\begin{align*}
\dot{z} &= rv \\
\dot{r} &= s\omega \\
\dot{s} &= -r\omega.
\end{align*}
\]

Let \( l > 0 \). The point \( p := z + lr \) is a distance \( l \) in front of the unicycle:

\[
\begin{align*}
\dot{p} &= rv + ls\omega.
\end{align*}
\]

Let \( u := rv + ls\omega \). Since \( r, s \) are orthonormal, the equation

\[
rv + ls\omega = u
\]

can be solved for \( v, \omega \), namely, \( v = u^T r, \omega = (1/l)(u^T s) \). Then the dynamics of the point \( p \) are simply \( \dot{p} = u \).

As an application, let's consider cyclic pursuit:

\[
\begin{align*}
\dot{z}_i &= r_i v_i \\
\dot{r}_i &= s_i \omega_i \\
\dot{s}_i &= -r_i \omega_i.
\end{align*}
\]

Define the just-ahead points \( p_i := z_i + lr_i \) and the pseudo-controls \( u_i := r_i v_i + ls_i \omega_i \). Then \( \dot{p}_i = u_i \). Cyclic pursuit for this system is

\[
u_i = p_{i+1} - p_i \pmod{n}.
\]

Then there is global convergence: \( p_i(t) \) all converge to the same point.

However, there are limitations of the approach. First, the controller is not allowable. Consider for example the control law for unicycle 1:

\[
u_1 = p_2 - p_1.
\]
Thus
\[ r_1 v_1 + l s_1 \omega_1 = z_2 + l r_2 - z_1 - l r_1. \]

I.e.,
\[ v_1 = (z_2 - z_1)^T r_1 + l (r_2^T r_1) - l \]
and
\[ \omega_1 = (1/l) (z_2 - z_1)^T s_1 + (r_2^T s_1). \]

The signals \((r_2^T r_1), (r_2^T s_1)\), i.e., the components of \(r_2\) in the basis \(\{r_1, s_1\}\), cannot be measured by a position sensor on unicycle 1.

Secondly, to get the unicycles almost to rendezvous, we would need \(l\) to be very small. But then the term \((1/l) (z_2 - z_1)^T s_1\) in \(\omega_1\) would be very large, saturating a real actuator.

Finally, the pseudo-linearization + cyclic pursuit approach can’t be considered an approximation of a direct cyclic pursuit strategy, because the qualitative behaviour of the latter is not recovered as \(l \to 0\).
5.3 Unicycles in Cyclic Pursuit

This section focusses on unicycles in cyclic pursuit, a specific distributed control strategy. We already studied point robots under cyclic pursuit. Would it be different for unicycles? What are achievable formations?

5.3.1 Redundant Models and Invariant Subspaces

There’s an important technical point that is best illustrated for point robots. So consider 3 point robots in cyclic pursuit in the plane:

\[
\begin{align*}
\dot{z}_1 &= u_1, \quad u_1 = z_2 - z_1 \\
\dot{z}_2 &= u_2, \quad u_2 = z_3 - z_2 \\
\dot{z}_3 &= u_3, \quad u_3 = z_1 - z_3.
\end{align*}
\]

These equations are in terms of the actual positions. Alternatively, since we’re interested in the formation, we might introduce relative coordinates, which in this case are just the errors

\[
e_1 = z_2 - z_1, \quad e_2 = z_3 - z_2, \quad e_3 = z_1 - z_3.
\]

Then for example

\[
\dot{e}_1 = \dot{z}_2 - \dot{z}_1 = (z_3 - z_2) - (z_2 - z_1) = e_2 - e_1.
\]

Thus we have

\[
\begin{align*}
\dot{e}_1 &= e_2 - e_1 \\
\dot{e}_2 &= e_3 - e_2 \\
\dot{e}_3 &= e_1 - e_3,
\end{align*}
\]

or \( \dot{e} = Ae \) where

\[
A = \begin{bmatrix}
-1 & 1 & 0 \\
0 & -1 & 1 \\
1 & 0 & -1
\end{bmatrix}.
\]

This model is redundant because the \( e_i \)'s are not independent—they satisfy the constraint

\[
e_1 + e_2 + e_3 = 0
\]

by their very definition. So the setup is

1. redundant model: \( \dot{e} = Ae \)

2. constraint equation: \( Ce = 0 \), i.e., \( e \in \ker C \), where \( C = 1^T \)

The subspace \( \ker C \) is \( A \)-invariant; indeed \( CA = 0 \). It follows that

\[
e(0) \in \ker C \implies (\forall t > 0)e(t) \in \ker C.
\]
In this example, a basis for $\ker C$ is
\[
v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.
\]
Complement these by adding, say,
\[
v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]
and then define
\[
V = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}.
\]
In this new basis $A$ becomes
\[
\tilde{A} = V^{-1}AV = \begin{bmatrix} -1 & 1 & 0 \\ -1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}.
\]
The upper-left block, $\begin{bmatrix} -1 & 1 \\ -1 & -2 \end{bmatrix}$, represents the restriction of $A$ to the invariant subspace $\ker C$ in the basis $\{v_1, v_2\}$; its eigenvalues are in the left half-plane. And the lower-right block, 0, represents $A$ in the factor space $\mathbb{R}^3/\ker C$. (We could change coordinates further to make the upper-right block zero.)

**Recap**

**Redundant model**
\[
\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}
\]

**Invariant constraint**
\[e_1 + e_2 + e_3 = 0\]
Change of coordinates—$\tilde{e} = V^{-1}e$:
\[
\tilde{e}_1 = e_1, \quad \tilde{e}_2 = e_2, \quad \tilde{e}_3 = e_1 + e_2 + e_3
\]

**Reduced model**
\[
\begin{bmatrix} \dot{\tilde{e}}_1 \\ \dot{\tilde{e}}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} \tilde{e}_1 \\ \tilde{e}_2 \end{bmatrix}
\]

So shouldn’t we use the reduced model? Well, there’s actually an advantage to using the redundant model—$A$ has a special structure that makes it easier to analyze: It’s a **circulant matrix**.
5.3.2 Circulant Matrices

Consider an ordered triple \((c_1, c_2, c_3)\) of real numbers. Now consider the two cyclic permutations of this triple by right-shift and wrapping:

\[
(c_3, c_1, c_2) \\
(c_2, c_3, c_1).
\]

Form the matrix with these three rows:

\[
C = \begin{bmatrix}
c_1 & c_2 & c_3 \\
c_3 & c_1 & c_2 \\
c_2 & c_3 & c_1
\end{bmatrix}.
\]

This is the general form of a \(3 \times 3\) circulant matrix. We’ll sometimes write the matrix as

\[
C = \text{circ}(c_1, c_2, c_3).
\]

It’s an important fact that every circulant matrix can be diagonalized, so its eigenvalues can be revealed. Begin with

\[
C = c_1 I + c_2 P + c_3 P^2 = q_C(P).
\]

By the \textbf{Spectral Mapping Theorem}, the eigenvalues of \(P\) are mapped under \(q_C\) to the eigenvalues of \(C\):

\[
eigs(C) = \{q_C(\lambda) : \lambda \text{ an eig of } P\}.
\]

Now \(P\) is a companion matrix and its characteristic polynomial is \(s^3 - 1\). So the eigenvalues of \(P\) are the three roots of unity:

\[1, e^{2\pi i/3}, e^{4\pi i/3}.
\]

Define \(\omega := e^{2\pi i/3}\), so that the three roots of unity are \(1, \omega, \omega^2\). Thus

\[
eigs(P) = \{1, \omega, \omega^2\}
\]
and
\[ \text{eigs}(C) = \{ q_C(1), q_C(\omega), q_C(\omega^2) \}. \]

Now let’s see how to diagonalize \( P \) and then \( C \). Since
\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix} =
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix},
\]
an eigenvector of \( P \) corresponding to the eigenvalue 1 is 1. Likewise, since \( \omega^3 = 1 \),
\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 \\
\omega \\
\omega^2
\end{bmatrix} = \omega
\begin{bmatrix}
1 \\
\omega \\
\omega^2
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 \\
\omega^2 \\
\omega^4
\end{bmatrix} = \omega^2
\begin{bmatrix}
1 \\
\omega^2 \\
\omega^4
\end{bmatrix}.
\]

Normalize these and define the matrices of eigenvalues and eigenvectors of \( P \):
\[
\Omega =
\begin{bmatrix}
1 & 0 & 0 \\
0 & \omega & 0 \\
0 & 0 & \omega^2
\end{bmatrix},
F = \frac{1}{\sqrt{3}}
\begin{bmatrix}
1 & 1 & 1 \\
1 & \omega & \omega^2 \\
1 & \omega^2 & \omega^4
\end{bmatrix}.
\]

This \( F \) comes up in the discrete Fourier transform. Thus
\[ PF = F\Omega. \]

The columns of \( F \) are actually orthonormal (proof?), so \( F \) is unitary and
\[ P = F\Omega F^*. \]

Now \( PF = F\Omega \) implies \( P^2F = F\Omega^2, P^3F = F\Omega^3 \), and so on. Since
\[ C = c_1I + c_2P + c_3P^2 = q_C(P), \]
we have
\[ CF = Fq_C(\Omega) \]
and hence
\[ C = F\Lambda F^*, \quad \Lambda := q_C(\Omega) =
\begin{bmatrix}
q_C(1) & 0 & 0 \\
0 & q_C(\omega) & 0 \\
0 & 0 & q_C(\omega^2)
\end{bmatrix}. \]
**Example** Let’s return to 3 point robots:

\[
\begin{align*}
\dot{z}_1 &= u_1, \quad u_1 = z_2 - z_1 \\
\dot{z}_2 &= u_2, \quad u_2 = z_3 - z_2 \\
\dot{z}_3 &= u_3, \quad u_3 = z_1 - z_3
\end{align*}
\]

\[
\dot{z} = Az
\]

\[
A = \begin{bmatrix}
-1 & 1 & 0 \\
0 & -1 & 1 \\
1 & 0 & -1
\end{bmatrix} = \text{circ}(-1, 1, 0)
\]

\[
\omega := e^{2\pi j/3}
\]

\[
F = \frac{1}{\sqrt{3}} \begin{bmatrix}
1 & 1 & \omega \\
1 & \omega & \omega^2 \\
1 & \omega^2 & \omega^4
\end{bmatrix}
\]

\[
q_A(s) = -1 + s
\]

\[
A = FAF^*
\]

\[
A = \begin{bmatrix}
q_A(1) & 0 & 0 \\
0 & q_A(\omega) & 0 \\
0 & 0 & q_A(\omega^2)
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 \\
0 & -1 + \omega & 0 \\
0 & 0 & -1 + \omega^2
\end{bmatrix}
\]

All of this extends to an \(n \times n\) circulant matrix \(C\).

### 5.3.3 Block Circulant Matrices

Bring in the Kronecker product of two matrices:

\[
A \otimes B = \begin{bmatrix}
a_{11}B & \cdots & a_{1n}B \\
\vdots & \ddots & \vdots \\
a_{m1}B & \cdots & a_{mn}B
\end{bmatrix}
\]

—replace the \(ij^{th}\) element of \(A\) by the block \(a_{ij}B\). E.g.,

\[
1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\
4 & 5 & 6 \end{bmatrix}, \quad 1 \otimes B = \begin{bmatrix} 1 & 2 & 3 \\
4 & 5 & 6 \\
1 & 2 & 3 \\
4 & 5 & 6
\end{bmatrix}
\]

\[
A = \begin{bmatrix} 1 & 2 \\
0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\
4 & 5 & 6 \end{bmatrix}, \quad A \otimes B = \begin{bmatrix} 1 & 2 & 3 & 2 & 4 & 6 \\
4 & 5 & 6 & 8 & 10 & 12 \\
0 & 0 & -1 & -2 & -3 \\
0 & 0 & -4 & -5 & -6
\end{bmatrix}
\]
Properties of the Kronecker product

KPa \((A + B) \otimes C = A \otimes C + B \otimes C\)

K Pb \((A \otimes B)^T = A^T \otimes B^T\)

K Pc \((A \otimes B)(C \otimes D) = (AC) \otimes (BD)\)

K Pd If \(A\) is square and \(B\) is square, the eigenvalues of \(A \otimes B\) are the products of the eigenvalues of \(A\) and of \(B\).

Now let \(A_1, A_2\) be square matrices—each, say, \(m \times m\). Then

\[
A = \begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix} =: \text{circ}(A_1, A_2)
\]

is an example of a block circulant matrix. We’ll block-diagonalize it. Define \(\omega_2 := e^{j\pi} = -1\), so that the two roots of unity are 1, \(\omega_2\). Bring in the Fourier matrix

\[
F_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & \omega_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.
\]

Then

\[
(F_2 \otimes I_m)^* A (F_2 \otimes I_m) = \frac{1}{2} \begin{bmatrix} I & I \\ I & -I \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix} \begin{bmatrix} I & I \\ I & -I \end{bmatrix}
\]

\[
= \frac{1}{2} \begin{bmatrix} I & I \\ I & -I \end{bmatrix} \begin{bmatrix} A_1 + A_2 & A_1 - A_2 \\ A_1 + A_2 & -A_1 + A_2 \end{bmatrix}
\]

\[
= \begin{bmatrix} A_1 + A_2 & 0 \\ 0 & A_1 - A_2 \end{bmatrix}.
\]

In other words

\[
(F_2 \otimes I_m)^* A (F_2 \otimes I_m) = \text{diag}(D_1, D_2),
\]

where

\[
\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \sqrt{2} (F_2 \otimes I_m) \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}.
\]

This construction generalizes to

\[
A = \text{circ}(A_1, A_2, \ldots, A_n).
\]

5.3.4 Cyclic Pursuit

Consider \(n > 1\) unicycles with unit speed:

\[
\begin{aligned}
\dot{x}_i &= \cos(\theta_i) \\
\dot{y}_i &= \sin(\theta_i) \\
\dot{\theta}_i &= \omega_i
\end{aligned}
\] or

\[
\begin{aligned}
\dot{z}_i &= r_i \\
r_i &= e^{j\theta_i} \\
\dot{\theta}_i &= \omega_i
\end{aligned}
\]
Introduce relative coordinates \((\rho_i, \alpha_i, \beta_i)\): \(\rho_i\) is defined as \(|z_{i+1} - z_i|\), and \(\alpha_i, \beta_i\) are the angles

Thus \((\rho_i, \alpha_i, \beta_i)\) are defined by two complex equations: Rotate \(r_i\) by \(\alpha_i\) and scale by \(\rho_i\) to get the relative displacement \(z_{i+1} - z_i\):

\[ z_{i+1} - z_i = \rho_i r_i e^{j\alpha_i}; \]

rotate \(r_{i+1}\) by \(\beta_i\) to get \(r_i\):

\[ r_{i+1} e^{j\beta_i} = r_i. \]

We now derive the differential equations in relative coordinates. From \(r_{i+1} e^{j\beta_i} = r_i\) we get

\[ \theta_i + \beta_i = \theta_i. \]  
\[ \dot{\beta}_i = \dot{\theta}_i = \omega_i - \omega_{i+1}. \] (5.1)

Differentiate \(z_{i+1} - z_i = \rho_i r_i e^{j\alpha_i}\):

\[ \dot{z}_{i+1} - \dot{z}_i = \dot{\rho}_i r_i e^{j\alpha_i} + \rho_i \dot{r}_i e^{j\alpha_i} + \rho_i r_i j \dot{e}^{j\alpha_i} \dot{\alpha}_i. \]

Sub for \(\dot{z}_{i+1}, \dot{z}_i, \dot{r}_i\):

\[ r_{i+1} - r_i = \dot{\rho}_i r_i e^{j\alpha_i} + \rho_i j r_i \omega_i e^{j\alpha_i} + \rho_i r_i j e^{j\alpha_i} \dot{\alpha}_i. \]

Sub for \(r_{i+1}\):

\[ r_i e^{-j\beta_i} - r_i = \dot{\rho}_i r_i e^{j\alpha_i} + \rho_i j r_i \omega_i e^{j\alpha_i} + \rho_i r_i j e^{j\alpha_i} \dot{\alpha}_i. \]

Cancel \(r_i\) and multiply by \(e^{-j\alpha_i}\):

\[ e^{-j(\alpha_i + \beta_i)} - e^{-j\alpha_i} = \dot{\rho}_i + \rho_i j \omega_i + \rho_i j \dot{\alpha}_i. \]

Take real parts:

\[ \cos(\alpha_i + \beta_i) - \cos(\alpha_i) = \dot{\rho}_i. \] (5.2)

Take imaginary parts:

\[ -\sin(\alpha_i + \beta_i) + \sin(\alpha_i) = \rho_i \omega_i + \rho_i \dot{\alpha}_i. \] (5.3)
Summary

Complex equations:
\[
\begin{align*}
\dot{\rho}_i + \rho_i j \omega_i + \rho_i j \dot{\alpha}_i &= e^{-j(\alpha_i + \beta_i)} - e^{-j\alpha_i} \\
\dot{\beta}_i &= \omega_i - \omega_{i+1}
\end{align*}
\]

Real equations:
\[
\begin{align*}
\dot{\rho}_i &= \cos(\alpha_i + \beta_i) - \cos(\alpha_i) \\
\dot{\alpha}_i &= \frac{1}{\rho_i} \{-\sin(\alpha_i + \beta_i) + \sin(\alpha_i)\} - \omega_i \\
\dot{\beta}_i &= \omega_i - \omega_{i+1}
\end{align*}
\]

Instead of designing control laws, we choose to study only the very simple control law
\[
\omega_i = k\alpha_i, \quad k > 0.
\]
That is, the turning velocity of each unicycle is proportional to the heading error. Thus we have the complex equations
\[
\begin{align*}
\dot{\rho}_i + \rho_i j k \alpha_i + \rho_i j \dot{\alpha}_i &= e^{-j(\alpha_i + \beta_i)} - e^{-j\alpha_i} \\
\dot{\beta}_i &= k(\alpha_i - \alpha_{i+1})
\end{align*}
\]
or the real equations
\[
\begin{align*}
\dot{\rho}_i &= \cos(\alpha_i + \beta_i) - \cos(\alpha_i) \\
\dot{\alpha}_i &= \frac{1}{\rho_i} \{-\sin(\alpha_i + \beta_i) + \sin(\alpha_i)\} - k\alpha_i \\
\dot{\beta}_i &= k(\alpha_i - \alpha_{i+1})
\end{align*}
\]

Equilibrium formations

Same concept as in Justh/Krishnaprasad, namely, the \( n \) unicycles are in an equilibrium formation if unicycle \( i + 1 \) appears stationary to unicycle \( i \). That is, there exist constants \( c_i, d_i \) such that
\[
z_{i+1} - z_i = c_i r_i, \quad r_{i+1} = d_i r_i.
\]
(5.4)

Since \( r_{i+1} e^{j\beta_i} = r_i \) and \( z_{i+1} - z_i = \rho_i r_i e^{j\alpha_i} \), this is equivalent to saying that
\[
\rho_i r_i e^{j\alpha_i} = c_i r_i, \quad r_{i+1} = d_i r_{i+1} e^{j\beta_i},
\]
i.e.,
\[
\rho_i e^{j\alpha_i} = c_i, \quad 1 = d_i e^{j\beta_i};
\]
equivalently, \( \rho_i, \alpha_i, \beta_i \) are constant.
5.3.5 Analysis of Two Unicycles

Obviously $\rho_1 = \rho_2$, but we're not going to invoke that fact. The redundant complex model is

\[
\begin{align*}
\dot{\rho}_1 + \rho_1 jk\alpha_1 + \rho_1 j\dot{\alpha}_1 &= e^{-j(\alpha_1 + \beta_1)} - e^{-j\alpha_1} \\
\dot{\beta}_1 &= k(\alpha_1 - \alpha_2) \\
\dot{\rho}_2 + \rho_2 jk\alpha_2 + \rho_2 j\dot{\alpha}_2 &= e^{-j(\alpha_2 + \beta_2)} - e^{-j\alpha_2} \\
\dot{\beta}_2 &= k(\alpha_2 - \alpha_1).
\end{align*}
\]

The redundant real model is

\[
\begin{align*}
\dot{\rho}_1 &= \cos(\alpha_1 + \beta_1) - \cos(\alpha_1) \\
\dot{\alpha}_1 &= \frac{1}{\rho_1}\{-\sin(\alpha_1 + \beta_1) + \sin(\alpha_1)\} - k\alpha_1 \\
\dot{\beta}_1 &= k(\alpha_1 - \alpha_2) \\
\dot{\rho}_2 &= \cos(\alpha_2 + \beta_2) - \cos(\alpha_2) \\
\dot{\alpha}_2 &= \frac{1}{\rho_2}\{-\sin(\alpha_2 + \beta_2) + \sin(\alpha_2)\} - k\alpha_2 \\
\dot{\beta}_2 &= k(\alpha_2 - \alpha_1).
\end{align*}
\]

The dimension is thus 6.

The two constraint equations are

\[(z_1 - z_2) + (z_2 - z_1) = 0\]
\[
\frac{r_2}{r_1} \frac{r_1}{r_2} = 1
\]

\[\Rightarrow\]
\[
\rho_2 e^{j\alpha_2} + \rho_1 e^{j\alpha_1} = 0 \\
e^{j\beta_2} e^{j\beta_1} = 1
\]

\[\Rightarrow\]
\[
\rho_2 e^{j\alpha_2} + \rho_1 e^{j(\beta_1 + \alpha_1)} = 0 \\
e^{j\beta_2} e^{j\beta_1} = 1
\]

\[\Rightarrow\]
\[
\rho_2 e^{j\alpha_2} + \rho_1 e^{j(\beta_1 + \alpha_1)} = 0 \\
e^{j(\beta_1 + \beta_2)} = 1
\]

\[\Rightarrow\]
\[
\rho_1 = \rho_2, \quad e^{j\alpha_2} = -e^{j(\beta_1 + \alpha_1)} \\
e^{j(\beta_1 + \beta_2)} = 1
\]
\[ \rho_1 = \rho_2 \]

\[ \beta_1 + \alpha_1 - \alpha_2 = \text{an odd multiple of } \pi \]

\[ \beta_1 + \beta_2 = \text{an even multiple of } \pi. \]

So we have 6 differential equations with 3 constraints, suggesting a state of dimension 3.

To find the equilibria it’s easier to work from the complex model:

\[
\begin{align*}
\dot{\rho}_1 + \rho_1 j k \alpha_1 + \rho_1 j \dot{\alpha}_1 &= e^{-j(\alpha_1 + \beta_1)} - e^{-j\alpha_1} \\
\dot{\beta}_1 &= k(\alpha_1 - \alpha_2) \\
\dot{\rho}_2 + \rho_2 j k \alpha_2 + \rho_2 j \dot{\alpha}_2 &= e^{-j(\alpha_2 + \beta_2)} - e^{-j\alpha_2} \\
\dot{\beta}_2 &= k(\alpha_2 - \alpha_1).
\end{align*}
\]

Set derivatives to 0:

\[
\begin{align*}
\rho_1 j k \alpha_1 &= e^{-j(\alpha_1 + \beta_1)} - e^{-j\alpha_1} \\
0 &= k(\alpha_1 - \alpha_2) \\
\rho_2 j k \alpha_2 &= e^{-j(\alpha_2 + \beta_2)} - e^{-j\alpha_2} \\
0 &= k(\alpha_2 - \alpha_1).
\end{align*}
\]

Thus \( \alpha_1 = \alpha_2 =: \alpha \). Now we invoke that \( \rho_1 = \rho_2 =: \rho \). And we consider only the case \( \rho > 0 \). So we have

\[
\begin{align*}
\rho j k \alpha &= e^{-j(\alpha + \beta_1)} - e^{-j\alpha} \\
\rho j k \alpha &= e^{-j(\alpha + \beta_2)} - e^{-j\alpha}.
\end{align*}
\]

Equivalently,

\[
\begin{align*}
\rho j k e^{j\alpha} &= e^{-j\beta_1} - 1 \\
\rho j k e^{j\alpha} &= e^{-j\beta_2} - 1.
\end{align*}
\]

This shows that \( e^{-j\beta_1} = e^{-j\beta_2} \).

Under the normalization \( \beta_i \in (-\pi, \pi] \), this implies

\( \beta_1 = \beta_2 =: \beta \).

Applying the constraint

\[ \rho_2 e^{j\alpha_2} + \rho_1 e^{j(\beta_1 + \alpha_1)} = 0, \]

we also have

\[ 1 + e^{j\beta} = 0. \]
Thus $\beta = \pi$. We had

$$\rho j k \alpha = e^{-j(\alpha + \beta)} - e^{-j\alpha},$$

so

$$\rho j k \alpha = -2e^{-j\alpha}.$$ 

This implies

$$\alpha = \pm \frac{\pi}{2}, \quad \rho k \frac{\pi}{2} = 2.$$ 

**In summary**, the equilibria are as follows:

<table>
<thead>
<tr>
<th>case 1</th>
<th>$\rho$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>case 2</td>
<td>$4/(\pi k)$</td>
<td>$\pi/2$</td>
<td>$\pi$</td>
</tr>
<tr>
<td>case 2</td>
<td>$4/(\pi k)$</td>
<td>$-\pi/2$</td>
<td>$\pi$</td>
</tr>
</tbody>
</table>

Circular motion.

**Local Stability Analysis**

Let's return to the redundant real model

$$\begin{align*}
\dot{\rho}_1 &= \cos(\alpha_1 + \beta_1) - \cos(\alpha_1) \\
\dot{\alpha}_1 &= \frac{1}{\rho_1} \{-\sin(\alpha_1 + \beta_1) + \sin(\alpha_1)\} - k\alpha_1 \\
\dot{\beta}_1 &= k(\alpha_1 - \alpha_2) \\
\dot{\rho}_2 &= \cos(\alpha_2 + \beta_2) - \cos(\alpha_2) \\
\dot{\alpha}_2 &= \frac{1}{\rho_2} \{-\sin(\alpha_2 + \beta_2) + \sin(\alpha_2)\} - k\alpha_2 \\
\dot{\beta}_2 &= k(\alpha_2 - \alpha_1).
\end{align*}$$

Coupling is through the $\dot{\beta}_1$ and $\dot{\beta}_2$ equations. The second constraint is

$$\beta_1 + \alpha_1 - \alpha_2 = \text{an odd multiple of } \pi,$$
which we can normalize to
\[ \beta_1 + \alpha_1 - \alpha_2 = \pi. \]

Using this, we get that the first three equations are
\[
\dot{\rho}_1 = \cos(\alpha_1 + \beta_1) - \cos(\alpha_1) \\
\dot{\alpha}_1 = \frac{1}{\rho_1} \{-\sin(\alpha_1 + \beta_1) + \sin(\alpha_1)\} - k\alpha_1 \\
\dot{\beta}_1 = -k\beta_1 + k\pi,
\]
which are self-contained. Dropping subscripts, we have that the problem reduces to the stability analysis of
\[
\dot{\rho} = \cos(\alpha + \beta) - \cos(\alpha) \\
\dot{\alpha} = \frac{1}{\rho} \{-\sin(\alpha + \beta) + \sin(\alpha)\} - k\alpha \\
\dot{\beta} = -k\beta + k\pi
\]
at the equilibria

<table>
<thead>
<tr>
<th>Case</th>
<th>(\rho)</th>
<th>(\alpha)</th>
<th>(\beta)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(4/(\pi k))</td>
<td>(\pi/2)</td>
<td>(\pi)</td>
</tr>
<tr>
<td>2</td>
<td>(4/(\pi k))</td>
<td>(-\pi/2)</td>
<td>(\pi)</td>
</tr>
</tbody>
</table>

The Jacobian is
\[
\begin{bmatrix}
0 & -\sin(\bar{\alpha} + \bar{\beta}) + \sin(\bar{\alpha}) & -\sin(\bar{\alpha} + \bar{\beta}) \\
-\frac{1}{\bar{\rho}} \{-\sin(\bar{\alpha} + \bar{\beta}) + \sin(\bar{\alpha})\} & \frac{1}{\bar{\rho}} \{-\cos(\bar{\alpha} + \bar{\beta}) + \cos(\bar{\alpha})\} - k & -\frac{1}{\bar{\rho}} \cos(\bar{\alpha} + \bar{\beta}) \\
0 & 0 & -k
\end{bmatrix}
\]

For the equilibrium \((\bar{\rho}, \bar{\alpha}, \bar{\beta}) = (4/(\pi k), \pi/2, \pi)\) the Jacobian is
\[
\begin{bmatrix}
0 & 2 & 1 \\
-\pi^2k^2/8 & -k & 0 \\
0 & 0 & -k
\end{bmatrix}
\]
locally asymptotically stable \(\forall k > 0\). For the equilibrium \((\bar{\rho}, \bar{\alpha}, \bar{\beta}) = (4/(\pi k), -\pi/2, \pi)\) the Jacobian is
\[
\begin{bmatrix}
0 & -2 & -1 \\
\pi^2k^2/8 & -k & 0 \\
0 & 0 & -k
\end{bmatrix}
\]
locally asymptotically stable \(\forall k > 0\).
5.3.6 Analysis of Three Unicycles

The redundant complex model is
\[\begin{align*}
\dot{\rho}_1 + \rho_1 jk\alpha_1 + \rho_1 j\dot{\alpha}_1 &= e^{-j(\alpha_1 + \beta_1)} - e^{-j\alpha_1} \\
\dot{\rho}_2 + \rho_2 jk\alpha_2 + \rho_2 j\dot{\alpha}_2 &= e^{-j(\alpha_2 + \beta_2)} - e^{-j\alpha_2} \\
\dot{\rho}_3 + \rho_3 jk\alpha_3 + \rho_3 j\dot{\alpha}_3 &= e^{-j(\alpha_3 + \beta_3)} - e^{-j\alpha_3} \\
\dot{\beta}_1 &= k(\alpha_1 - \alpha_2) \\
\dot{\beta}_2 &= k(\alpha_2 - \alpha_3) \\
\dot{\beta}_3 &= k(\alpha_3 - \alpha_1).
\end{align*}\]

The redundant real model is
\[\begin{align*}
\dot{\rho}_1 &= \cos(\alpha_1 + \beta_1) - \cos(\alpha_1) \\
\dot{\alpha}_1 &= \frac{1}{\rho_1}\{-\sin(\alpha_1 + \beta_1) + \sin(\alpha_1)\} - k\alpha_1 \\
\dot{\beta}_1 &= k(\alpha_1 - \alpha_2) \\
\dot{\rho}_2 &= \cos(\alpha_2 + \beta_2) - \cos(\alpha_2) \\
\dot{\alpha}_2 &= \frac{1}{\rho_2}\{-\sin(\alpha_2 + \beta_2) + \sin(\alpha_2)\} - k\alpha_2 \\
\dot{\beta}_2 &= k(\alpha_2 - \alpha_3) \\
\dot{\rho}_3 &= \cos(\alpha_3 + \beta_3) - \cos(\alpha_3) \\
\dot{\alpha}_3 &= \frac{1}{\rho_3}\{-\sin(\alpha_3 + \beta_3) + \sin(\alpha_3)\} - k\alpha_3 \\
\dot{\beta}_3 &= k(\alpha_3 - \alpha_1).
\end{align*}\]

The dimension is thus 9. The structure as a block diagram:

The two constraint equations are
\[\begin{align*}
(z_1 - z_3) + (z_3 - z_2) + (z_2 - z_1) &= 0 \\
\frac{r_3}{r_1} \frac{r_2}{r_3} &= 1 \\
\Rightarrow \quad \rho_3 r_3 e^{j\alpha_3} + \rho_2 r_2 e^{j\alpha_2} + \rho_1 r_1 e^{j\alpha_1} &= 0
\end{align*}\]
\[ e^{j\beta_3} e^{j\beta_2} e^{j\beta_1} = 1 \]

\[ \Rightarrow \]

\[ \rho_3 r_3 e^{j\alpha_1} + \rho_2 r_3 e^{j\beta_1} e^{j\alpha_2} + \rho_1 r_3 e^{j\beta_1} e^{j\alpha_1} = 0 \]

\[ e^{j\beta_3} e^{j\beta_2} e^{j\beta_1} = 1 \]

\[ \Rightarrow \]

\[ \rho_3 r_3 e^{j\alpha_3} + \rho_2 r_3 e^{j(\beta_1 + \alpha_2)} + \rho_1 r_3 e^{j(\beta_1 + \alpha_1)} = 0 \]

\[ e^{j(\beta_1 + \beta_2 + \beta_3)} = 1 \]

\[ \Rightarrow \]

\[ \rho_3 \cos(\alpha_3) + \rho_2 \cos(\beta_1 + \alpha_2) + \rho_1 \cos(\beta_1 + \beta_1 + \alpha_1) = 0 \]

\[ \rho_3 \sin(\alpha_3) + \rho_2 \sin(\beta_1 + \alpha_2) + \rho_1 \sin(\beta_1 + \beta_1 + \alpha_1) = 0 \]

\[ \beta_3 + \beta_2 + \beta_1 = \text{an even multiple of } \pi. \]

So we have 9 differential equations with 3 constraints, suggesting a state of dimension 6.

For the equilibria, set the derivatives to zero in the 6 complex equations:

\[
\begin{align*}
\rho_1 j k \alpha_1 &= e^{-j(\alpha_1 + \beta_1)} - e^{-j\alpha_1} \\
\rho_2 j k \alpha_2 &= e^{-j(\alpha_2 + \beta_2)} - e^{-j\alpha_2} \\
\rho_3 j k \alpha_3 &= e^{-j(\alpha_3 + \beta_3)} - e^{-j\alpha_3}
\end{align*}
\]

\[
\begin{align*}
0 &= k(\alpha_1 - \alpha_2) \\
0 &= k(\alpha_2 - \alpha_3) \\
0 &= k(\alpha_3 - \alpha_1).
\end{align*}
\]

So

\[ \alpha_1 = \alpha_2 = \alpha_3 =: \alpha \]

and then

\[
\begin{align*}
\rho_1 j k \alpha &= e^{-j(\alpha + \beta_1)} - e^{-j\alpha} \\
\rho_2 j k \alpha &= e^{-j(\alpha + \beta_2)} - e^{-j\alpha} \\
\rho_3 j k \alpha &= e^{-j(\alpha + \beta_3)} - e^{-j\alpha}.
\end{align*}
\]
Equivalently,
\[
\begin{align*}
\rho_1 jk\alpha e^{i\alpha} & = e^{-j\beta_1} - 1 \\
\rho_2 jk\alpha e^{i\alpha} & = e^{-j\beta_2} - 1 \\
\rho_3 jk\alpha e^{i\alpha} & = e^{-j\beta_3} - 1.
\end{align*}
\]
This shows that
\[
e^{-j\beta_1} - 1, e^{-j\beta_2} - 1, e^{-j\beta_3} - 1
\]
all have the same angle. Under the normalization \(\beta_i \in (-\pi, \pi]\), this implies
\[
\beta_1 = \beta_2 = \beta_3 =: \beta.
\]
Then from
\[
\begin{align*}
\rho_1 jk\alpha e^{i\alpha} & = e^{-j\beta} - 1 \\
\rho_2 jk\alpha e^{i\alpha} & = e^{-j\beta} - 1 \\
\rho_3 jk\alpha e^{i\alpha} & = e^{-j\beta} - 1,
\end{align*}
we get
\[
\rho_1 = \rho_2 = \rho_3 =: \rho.
\]
Applying the constraint
\[
\rho_3 e^{j\alpha_3} + \rho_2 e^{j(\beta_1 + \alpha_2)} + \rho_1 e^{j(\beta_2 + \beta_1 + \alpha_1)} = 0
\]
we also have
\[
1 + e^{j\beta} + e^{j2\beta} = 0.
\]
Since
\[
\frac{z^3 - 1}{z - 1} = z^2 + z + 1,
\]
we see that \(e^{j\beta}\) is one of the third-roots of 1, but not 1. Thus \(\beta = \pm 2\pi/3\).
We have
\[
\rho jk\alpha e^{j\alpha} = e^{-j\beta} - 1.
\]
If \(\beta = 2\pi/3\), then
\[
\rho jk\alpha e^{j\alpha} = e^{-j2\pi/3} - 1 = \sqrt{3}e^{j5\pi/6}.
\]
So, if \(\alpha > 0\),
\[
\alpha + \pi/2 = 5\pi/6, \quad \rho k\alpha = \sqrt{3},
\]

\[ \alpha = \pi/3, \quad \rho = 3\sqrt{3}/(\pi k); \]

if \( \alpha < 0, \)

\[ \alpha + \pi/2 + \pi = 5\pi/6, \quad \rho k|\alpha| = \sqrt{3}, \]

i.e.,

\[ \alpha = -2\pi/3, \quad \rho = 3\sqrt{3}/(2\pi k). \]

If \( \beta = -2\pi/3, \) then

\[ \rho j k \alpha e^{j\alpha} = e^{j2\pi/3} - 1 = \sqrt{3}e^{-j5\pi/6}. \]

So, if \( \alpha > 0, \)

\[ \alpha + \pi/2 = -5\pi/6, \quad \rho k \alpha = \sqrt{3}, \]

i.e.,

\[ \alpha = 2\pi/3, \quad \rho = 3\sqrt{3}/(2\pi k); \]

if \( \alpha < 0, \)

\[ \alpha + \pi/2 + \pi = -5\pi/6, \quad \rho k|\alpha| = \sqrt{3}, \]

i.e.,

\[ \alpha = -\pi/3, \quad \rho = 3\sqrt{3}/(\pi k). \]

In summary, the equilibria are as follows:

<table>
<thead>
<tr>
<th></th>
<th>( \rho )</th>
<th>( \alpha )</th>
<th>( \beta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>case 1</td>
<td>3\sqrt{3}/(\pi k)</td>
<td>\pi/3</td>
<td>-2\pi/3</td>
</tr>
<tr>
<td>case 2</td>
<td>3\sqrt{3}/(2\pi k)</td>
<td>2\pi/3</td>
<td>2\pi/3</td>
</tr>
<tr>
<td>case 3</td>
<td>3\sqrt{3}/(\pi k)</td>
<td>-\pi/3</td>
<td>2\pi/3</td>
</tr>
<tr>
<td>case 4</td>
<td>3\sqrt{3}/(2\pi k)</td>
<td>-2\pi/3</td>
<td>-2\pi/3</td>
</tr>
</tbody>
</table>

The four equilibrium formations are as follows:
These four equilibrium formations can be described as follows:

- The unicycles are on a circle and their order (1,2,3) is either clockwise (cases 2,3) or counterclockwise (cases 1,4).

- The unicycles form an equilateral triangle.

- The rotation of the circle is either clockwise (cases 3,4) or counterclockwise (cases 1,2).

5.3.7 n Unicycles

As we’ve seen for \( n = 2, 3 \), in equilibrium

\[
\rho_1 = \cdots = \rho_n =: \rho \\
\alpha_1 = \cdots = \alpha_n =: \alpha \\
\beta_1 = \cdots = \beta_n =: \beta.
\]

So let’s assume this. What are the possible values of \( \rho, \alpha, \beta \)? Let \( z = e^{j\beta} \), so \( r_{i+1}z = r_i \). The constraint

\[
(z_1 - z_n) + (z_n - z_{n-1}) + \cdots + (z_2 - z_1) = 0
\]

implies

\[
pr_n e^{j\alpha} + \cdots + pr_1 e^{j\alpha} = 0.
\]

That is

\[
r_n + \cdots + r_1 = 0 : \quad (5.5)
\]
In equilibrium, the centroid of the velocity vectors is the origin. Using $r_{i+1}z = r_i$, we get

$$r_n + \cdots + r_3 + r_2 + r_2z = 0,$$
$$r_n + \cdots + r_3 + r_3z + r_3z^2 = 0,$$

and so on until

$$r_n(1 + z + \cdots + z^{n-1}) = 0.$$ 

Cancel $r_n$:

$$\frac{1 - z^n}{1 - z} = 1 + z + \cdots + z^{n-1} = 0.$$ 

Thus $z = e^{j\beta}$ is an $n^{th}$-root of unity but not 1. There are therefore $n - 1$ equilibrium values of $\beta$, given by

$$\frac{2\pi}{n} p, \quad p = 1, \ldots, n - 1$$

if the range is taken as $(0, 2\pi]$. They could be written over the range $(-\pi, \pi]$: If $n$ is odd, the values of $\beta$ are

$$\pm \frac{2\pi}{n} p, \quad p = 1, \ldots, (n - 1)/2;$$

and if $n$ is even, the values of $\beta$ are

$$\pi, \pm \frac{2\pi}{n} p, \quad p = 1, \ldots, (n - 2)/2.$$ 

Then $\rho, \alpha$ satisfy

$$\rho jke^{jx}e^{j\alpha} = e^{-j\beta} - 1.$$ 

For each $\beta$, there are two values of $\alpha$—one positive and one negative—and one value of $\rho$. So altogether there are $2(n - 1)$ equilibria in $(\rho, \alpha, \beta)$-space, $\mathbb{R}^3$.

Formulae for $\rho, \alpha$ can be found as follows. First, suppose $n$ is odd, $\beta$ positive:

$$\frac{2\pi}{n} p, \quad p = 1, \ldots, (n - 1)/2.$$ 

![Diagram](image-url)

length $= 2 \sin(\beta/2)$
Thus
\[ e^{-j\beta} - 1 = 2\sin\left(\frac{\beta}{2}\right) e^{j\left(\frac{\pi - \beta}{2}\right)} = -2\sin\left(\frac{\beta}{2}\right) e^{j\left(\frac{\pi - \beta}{2}\right)}. \]

So
\[ \rho j k \alpha e^{j\alpha} = -2\sin\left(\frac{\beta}{2}\right) e^{j\left(\frac{\pi - \beta}{2}\right)} \]
and hence
\[ \rho k \alpha e^{j\alpha} = -2\sin\left(\frac{\beta}{2}\right) e^{j\left(-\frac{\beta}{2}\right)}. \]

The positive \( \alpha \) satisfies
\[ \rho k \alpha e^{j\alpha} = 2\sin\left(\frac{\beta}{2}\right) e^{j\left(\pi - \frac{\beta}{2}\right)}, \]
i.e.,
\[ \alpha = \pi - \frac{\beta}{2}, \]
and the negative \( \alpha \) satisfies
\[ \rho k |\alpha| e^{j\alpha} = 2\sin\left(\frac{\beta}{2}\right) e^{j\left(-\frac{\beta}{2}\right)}, \]
i.e.,
\[ \alpha = -\frac{\beta}{2}. \]

The case \( \beta < 0 \) is similar.

In summary, for \( n \) odd, the values of \( \beta \) are
\[ \pm \frac{2\pi}{n} p, \quad p = 1, \ldots, (n - 1)/2, \]
and for each of these values there are two values of \( \alpha \), which can be read from the graph (slope = \(-1/2\)):
For $n$ even, the values of $\beta$ are
\[ \pi, \pm \frac{2\pi}{n} p, \quad p = 1, \ldots, (n-2)/2, \]
and for each of these values there are two values of $\alpha$, which can be read from the same graph. In all cases, the formula for $\rho$ is
\[ \rho = \frac{2}{k|\alpha|} \sin \left( \frac{|\beta|}{2} \right). \]

**Generalized Regular Polygon**

To describe the equilibrium formations geometrically, we introduce the notion of a generalized regular polygon. Let $n$ and $d$ be positive integers, $d < n$. Let $R$ be the positive rotation in $\mathbb{C}$, about the origin, through the angle $2\pi d/n$. Define vertices
\[ z_1 \neq 0; \quad z_{i+1} = Rz_i, \quad i = 1, \ldots, n-1 \]
and directed edges
\[ e_i: z_i \rightarrow z_{i+1}, \quad i = 1, \ldots, n. \]

The resulting figure is a generalized regular polygon, denoted $\Pi_{n,d}$. E.g., $n = 3$

E.g., $n = 4$:
E.g., \( n = 5 \):

Thus \( \Pi_{n,d} \) is a directed graph with vertices \( z_i \), not necessarily distinct, connected by edges \( e_i \). When \( d = 1 \), we have an ordinary regular polygon, e.g., \( \Pi_{5,1} \). When \( n, d \) are coprime, we have a regular polygon. Sometimes we have a star polygon, e.g., \( \Pi_{5,2} \). If \( n \) and \( d \) have a greatest common factor \( m > 1 \), then \( \Pi_{n,d} \) has \( n/m \) distinct vertices and \( n/m \) edges traversed \( m \) times, e.g., \( \Pi_{4,2} \).

Each equilibrium formation can be regarded as a directed graph: edge from \( i \) to \( i + 1 \). Let us consider counterclockwise motions only. There are then \( n - 1 \) distinct equilibrium formations. Likewise, there are \( n - 1 \) generalized regular polygons on \( n \) vertices, \( \Pi_{n,d}, d = 1, \ldots, n - 1 \).

**Theorem 30** Each equilibrium formation is a generalized regular polygon, and vice versa. In particular, in an equilibrium formation, the unicycles lie on a circle with stationary centre.

**Proof** For clarity, let’s do the proof for \( n = 5 \).

There are \( 2(n - 1) = 8 \) equilibrium formations—half rotate clockwise, half counterclockwise. We’ll focus only on the 4 counterclockwise ones, for which \( \alpha > 0 \). The equilibrium values of \( \alpha, \beta \) are

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>(-\frac{4}{5}\pi)</th>
<th>(-\frac{2}{5}\pi)</th>
<th>(\frac{2}{5}\pi)</th>
<th>(\frac{4}{5}\pi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td>(\frac{2}{5}\pi)</td>
<td>(\frac{1}{5}\pi)</td>
<td>(\frac{4}{5}\pi)</td>
<td>(\frac{3}{5}\pi)</td>
</tr>
</tbody>
</table>

The formations are as follows.

\( \alpha = \pi/5, \beta = -2\pi/5 \)
Notice that the velocity vectors are tangent to the circle through the vertices. The generalized regular polygon is $\Pi_{5,1}$.

$$\alpha = 2\pi/5, \beta = -4\pi/5$$

The generalized regular polygon is $\Pi_{5,2}$.

$$\alpha = 3\pi/5, \beta = 4\pi/5$$
The generalized regular polygon is $\Pi_{5,3}$.

$\alpha = 4\pi/5, \beta = 2\pi/5$

The generalized regular polygon is $\Pi_{5,4}$.

Finally, since in general $z_i = r_i$, the centroid, $z_c$, of the unicycles’ locations satisfies

$$\dot{z}_c = \frac{1}{n}(\dot{z}_1 + \cdots + \dot{z}_n) = \frac{1}{n}(r_1 + \cdots + r_n).$$

But we saw, (5.5), that at equilibrium

$$r_1 + \cdots + r_n = 0.$$ 

Thus $z_c$ is a constant at equilibrium. $\square$
Local Stability

We now study stability of the equilibrium formations. Again, by symmetry it suffices to handle the counterclockwise motions.

**Theorem 31** The following table shows which $\Pi_{n,d}$ are locally asymptotically stable for $n \geq 2$. Stability is independent of $k > 0$.

<table>
<thead>
<tr>
<th>$d$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>all</td>
</tr>
<tr>
<td>2</td>
<td>$\geq 4$</td>
</tr>
<tr>
<td>3</td>
<td>$\geq 10$</td>
</tr>
<tr>
<td>4</td>
<td>$\geq 21$</td>
</tr>
<tr>
<td>5</td>
<td>$\geq 54$</td>
</tr>
<tr>
<td>$\geq 6$</td>
<td>none</td>
</tr>
</tbody>
</table>

We are going to develop the proof technique by revisiting the case $n = 2$, which we analyzed before by simply decoupling the two unicycles.

**Proof Technique, $n = 2$**

The redundant real model is

- $\dot{\rho}_1 = \cos(\alpha_1 + \beta_1) - \cos(\alpha_1)$
- $\dot{\alpha}_1 = \frac{1}{\rho_1} \{- \sin(\alpha_1 + \beta_1) + \sin(\alpha_1)\} - k\alpha_1$
- $\dot{\beta}_1 = k(\alpha_1 - \alpha_2)$
- $\dot{\rho}_2 = \cos(\alpha_2 + \beta_2) - \cos(\alpha_2)$
- $\dot{\alpha}_2 = \frac{1}{\rho_2} \{- \sin(\alpha_2 + \beta_2) + \sin(\alpha_2)\} - k\alpha_2$
- $\dot{\beta}_2 = k(\alpha_2 - \alpha_1)$.

Define the state vector of unicycle $i$:

$$\xi_i = (\rho_i, \alpha_i, \beta_i).$$

Then the above nonlinear equations can be written as

$$\dot{\xi}_1 = f(\xi_1, \xi_2), \quad \dot{\xi}_2 = f(\xi_2, \xi_1),$$

(note that $f$ doesn’t depend on $i$) and these can be assembled as

$$\dot{\xi} = h(\xi),$$

where

$$\xi = (\xi_1, \xi_2) \in \mathbb{R}^6.$$

As we know, the equation $\dot{\xi} = h(\xi)$ is redundant: The two constraint equations

$$(z_1 - z_2) + (z_2 - z_1) = 0$$
\[
\frac{r_2}{r_1} = \frac{r_1}{r_2} = 1
\]

\[\Rightarrow\]

\[
\rho_2 r_2 e^{i\alpha_2} + \rho_1 r_1 e^{i\alpha_1} = 0
\]

\[
e^{i\beta_2} e^{i\beta_1} = 1
\]

\[\Rightarrow\]

\[
\rho_2 e^{i\alpha_2} + \rho_1 e^{i(\beta_1 + \alpha_1)} = 0
\]

\[
e^{i(\beta_1 + \beta_2)} = 1
\]

The latter two equations have the form

\[g(\xi) = 0,\]

where \(g: \mathbb{R}^6 \rightarrow \mathbb{C}^2:\)

\[
g(\xi) = \begin{bmatrix}
\rho_2 e^{i\alpha_2} + \rho_1 e^{i(\beta_1 + \alpha_1)} \\
e^{i(\beta_1 + \beta_2)} - 1
\end{bmatrix}.
\]

(It’s convenient to keep \(g\) complex-valued.) Thus the model with constraint is

\[\dot{\xi} = h(\xi)
\]

\[g(\xi) = 0.\]

There is one equilibrium formation (counting counterclockwise ones only). Denote it by

\[(\bar{\rho}, \bar{\alpha}, \bar{\beta})\]

and define

\[\bar{\xi}_i = (\bar{\rho}, \bar{\alpha}, \bar{\beta}), \quad i = 1, 2\]

and

\[\bar{\xi} = (\bar{\xi}_1, \bar{\xi}_2)\]

Then \(h(\bar{\xi}) = 0\) and \(g(\bar{\xi}) = 0\). The linearization of

\[\dot{\xi} = h(\xi)
\]

\[g(\xi) = 0\]

at the equilibrium \(\bar{\xi}\) is

\[\Delta \dot{\xi} = A \Delta \xi
\]

\[C \Delta \xi = 0\]

\[A = \frac{\partial h}{\partial \xi}(\bar{\xi}), \quad C = \frac{\partial g}{\partial \xi}(\bar{\xi}).\]
Let’s take (case 1) $\bar{\rho} = 4/\pi k, \bar{\alpha} = \pi/2, \bar{\beta} = \pi$. We get

$$A = \text{circ}(A_1, A_2)$$

$$A_1 = \begin{bmatrix} 0 & 2 & 1 \\ -\pi^2 k^2 / 8 & -k & 0 \\ 0 & k & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -k & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} -j & \bar{\rho} & \bar{\rho} & j & -\bar{\rho} & 0 \\ 0 & 0 & j & 0 & 0 & j \end{bmatrix}.$$ 

This $C$ has the same kernel in $\mathbb{R}^6$ as the real matrix

$$C = \begin{bmatrix} 0 & 1 & 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix},$$

so let’s use the latter. Partition $C$ as

$$C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}.$$ 

Then $\ker C$ is spanned by the columns of

$$\begin{bmatrix} I \\ -T \end{bmatrix}, \quad T = C_2^{-1}C_1.$$ 

Transform $A$ via

$$\begin{bmatrix} I & 0 \\ -T & I \end{bmatrix};$$

$$\begin{bmatrix} I & 0 \\ T & I \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix} \begin{bmatrix} I & 0 \\ -T & I \end{bmatrix} = \begin{bmatrix} A_1 - A_2T & A_2 \\ 0 & A_1 + TA_2 \end{bmatrix}.$$ 

You may verify that the lower-left block in the previous matrix equals 0, because $\ker C$ is $A$-invariant. Thus $A_1 - A_2T$ is a matrix representation of the restriction of $A$ to $\ker C$, and $A_1 + TA_2$ represents the projection of $A$ in the quotient space $\mathbb{R}^6/\ker C$. Here

$$A_1 + TA_2 = \begin{bmatrix} 0 & 2 & 1 \\ -\pi^2 k^2 / 8 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

whose eigenvalues are $0, \pm \frac{\pi k}{2} j$.

**Recap**  We’re studying 2 unicycles. The linearized model is

$$\dot{\Delta \xi} = A \Delta \xi, \quad C \Delta \xi = 0,$$

where $A$ is a $6 \times 6$ circulant matrix and $C$ is $3 \times 6$, rank 3. The system evolves on the $A$-invariant subspace $\ker C$. The eigenvalues of the projection of $A$ in the quotient space $\mathbb{R}^6/\ker C$ are $0, \pm \frac{\pi k}{2} j$. Thus the eigenvalues of the restriction of $A$ to $\ker C$ are the eigenvalues of $A$ other than these.
Therefore, if we can show the remaining eigenvalues of $A$ lie in $\text{Re} \ s < 0$, then we’ve proved local asymptotic stability of the equilibrium formation.

As we saw before, the matrix $\text{circ}(A_1, A_2)$ can be block-diagonalized to

$$\text{diag}(A_1 + A_2, A_1 - A_2).$$

Here,

$$A_1 + A_2 = \begin{bmatrix} 0 & 2 & 1 \\ -\pi^2k^2/8 & -k & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_1 - A_2 = \begin{bmatrix} 0 & 2 & 1 \\ -\pi^2k^2/8 & -k & 0 \\ 0 & 2k & 0 \end{bmatrix}. $$

The characteristic polynomial of $A_1 + A_2$ is

$$s \left( s^2 + ks + \frac{\pi^2k^2}{4} \right),$$

whose roots are in $\text{Re} \ s < 0$ except for the one at $s = 0$. The characteristic polynomial of $A_1 - A_2$ is

$$(s + k) \left( s^2 + \frac{\pi^2k^2}{4} \right),$$

whose roots are in $\text{Re} \ s < 0$ except for the two imaginary ones. We conclude that the equilibrium formation is locally asymptotically stable.

The other entries in the table require an ingenious proof that we don’t have time to cover.

### 5.3.8 Exercises

1. Let $A_1, A_2, A_3$ be square matrices—each, say, $m \times m$. Then

$$A = \begin{bmatrix} A_1 & A_2 & A_3 \\ A_3 & A_1 & A_2 \\ A_2 & A_3 & A_1 \end{bmatrix} =: \text{circ}(A_1, A_2, A_3)$$

is an example of a block circulant matrix. To block-diagonalize it, define $\omega_3 := e^{2\pi j/3}$, so that the three roots of unity are $1, \omega_3, \omega_3^2$. Bring in the Fourier matrix

$$F_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega_3 & \omega_3^2 \\ 1 & \omega_3^2 & \omega_3 \end{bmatrix}. $$

Derive that

$$(F_3 \otimes I_m) A (F_3 \otimes I_m) = \frac{1}{3} \begin{bmatrix} I & I & I \\ I & \omega_3 I & \omega_3^2 I \\ I & \omega_3^2 I & \omega_3 I \end{bmatrix} \begin{bmatrix} A_1 & A_2 & A_3 \\ A_3 & A_1 & A_2 \\ A_2 & A_3 & A_1 \end{bmatrix} \begin{bmatrix} I & I & I \\ I & \omega_3 I & \omega_3^2 I \\ I & \omega_3^2 I & \omega_3 I \end{bmatrix} \begin{bmatrix} A_1 + A_2 + A_3 & 0 & 0 \\ 0 & A_1 + \omega_3 A_2 + \omega_3^2 A_3 & 0 \\ 0 & 0 & A_1 + \omega_3^2 A_2 + \omega_3 A_3 \end{bmatrix}. $$
In other words

\[(F_3 \otimes I_m)^* A (F_3 \otimes I_m) = \text{diag}(D_1, D_2, D_3),\]

where

\[
\begin{bmatrix}
D_1 \\
D_2 \\
D_3
\end{bmatrix} = \sqrt{3} (F_3 \otimes I_m) \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}.
\]

2. Find all the equilibrium values of \((\rho, \alpha, \beta)\) for \(n = 4\). Sketch the corresponding equilibrium configurations of the unicycles. Repeat for \(n = 5\).

3. Draw \(\Pi_{6,d}, d = 1, \ldots, 5\) and \(\Pi_{7,d}, d = 1, \ldots, 6\).

4. The radius of the circle on which the unicycles travel depends on the designable parameter \(k > 0\) and other constants. Find a formula for the radius.

5. Prove that \(\Pi_{3,2}\) is not locally asymptotically stable.

6. Consider a mixture of point robots and unicycles in cyclic pursuit (heterogeneous network). Find equilibrium formations. Study their stability.

5.3.9 References

5.4 The Rendezvous Problem for Unicycles

5.4.1 Review

Let’s recall our results for \( n > 1 \) point robots modeled by complex numbers, \( z_1, \ldots, z_n \), in the plane. Velocity control: \( \dot{z}_i = u_i \). Each point robot senses the relative positions of a subgroup, \( \mathcal{N}_i \), of the others. Let \( y_i \) denote the vector with components \( z_m - z_i, m \in \mathcal{N}_i \). Thus \( y_i \) represents the information available to \( u_i \). Controllers are of the form \( u_i = F_i y_i \), or \( u_i = 0 \) if \( \mathcal{N}_i \) is empty.

Property: \( y_i = 0 \) or empty \( \implies u_i = 0 \implies \dot{z}_i = 0 \).

The problem was to find, if possible, \( F_1, \ldots, F_n \) such that

\[
(\forall \, z(0))(\exists \, z_{ss}) \lim_{t \to \infty} z(t) = z_{ss}1.
\]

Define the sensor graph \( \mathcal{G} \): There is an arc from \( i \) to \( m \) if and only if \( m \in \mathcal{N}_i \).

**Theorem 32**

1. Problem is solvable if and only if \( \mathcal{G} \) has a globally reachable node.

2. When the problem is solvable, one solution is

\[
F_i = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}.
\]

The goal in this chapter is to extend this result to unicycles.

5.4.2 Setup

Consider \( n > 1 \) unicycles:

\[
\begin{aligned}
\begin{cases}
\dot{x}_i = v_i \cos(\theta_i) \\
\dot{y}_i = v_i \sin(\theta_i) \\
\dot{\theta}_i = \omega_i
\end{cases}
\end{aligned}
\]

or

\[
\begin{aligned}
\begin{cases}
\dot{z}_i = v_i e^{i \theta_i} \\
\dot{\theta}_i = \omega_i
\end{cases}
\end{aligned}
\]

or

\[
\begin{aligned}
\begin{cases}
\dot{z}_i = r_i v_i \\
\dot{r}_i = s_i \omega_i \\
\dot{s}_i = -r_i \omega_i
\end{cases}
\end{aligned}
\]

Unicycle \( i \) can measure only the relative positions of sensed vehicles with respect to its own Frenet-Serret frame. Suppose unicycle \( i \) can sense unicycle \( m \). Write the relative position \( z_m - z_i \) in \( i \)'s frame:

\[
z_m - z_i = x_{im} r_i + y_{im} s_i.
\]
Thus we assume unicycle $i$ can measure $x_{im}, y_{im}$ about unicycle $m$. Note that
\[ x_{im} = (z_m - z_i)^T r_i, \]
\[ y_{im} = (z_m - z_i)^T s_i. \]
Again $N_i$ denotes those vehicles sensed by unicycle $i$.

An allowable controller looks like
\[ v_i = g_i(t, \{x_{im}, y_{im}\}_{m \in N_i}), \quad \omega_i = h_i(t, \{x_{im}, y_{im}\}_{m \in N_i}), \]
where $g_i, h_i$ are smooth functions of their arguments, and $g_i$ is such that
\[ \{N_i = \phi\} \Rightarrow \{v_i = 0\}, \]
i.e., there is zero forward velocity when the unicycle cannot sense any other vehicle, and
\[ \{(\forall m \in N_i) \ z_m = z_i\} \Rightarrow \{v_i = 0\}, \]
i.e., there is zero forward velocity when the unicycle’s neighbors have all converged to its position.

**Problem 1** Find, if possible, an allowable controller such that
\[ (\forall t_0)(\forall x_i(t_0), y_i(t_0), \theta_i(t_0); i = 1, \ldots, n)(\exists z_{ss} \in \mathbb{R}^2)(\forall i) \]
\[ \lim_{t \to \infty} z_i(t) = z_{ss}. \]

Bring in the sensor digraph $G$.

**Theorem 33** Problem 1 is solvable if and only if $G$ has a globally reachable node.

So the solvability condition for rendezvous of unicycles is exactly the same as for point robots!

**Proof of Theorem 33** ($\implies$) Same as for point robots. Fix any allowable controller. If $G$ does not have a globally reachable node, then there are two nonempty disjoint closed sets of nodes—say, $N_1$ and $N_2$. Start all the unicycles in $N_1$ at some initial condition $z_i(0) = a_1$ and all the unicycles in $N_2$ at some other initial condition $z_i(0) = a_2$. These two groups of unicycles will not move in the plane (though they may rotate), and hence convergence to a common $z_{ss}$ does not occur.

The proof of ($\impliedby$) is constructive and requires a lot of work.

**5.4.3 Proof of Sufficiency**

The controller is defined as follows. If $N_i = \phi$, i.e., unicycle $i$ senses no other unicycle, then
\[ \quad v_i(t) = 0, \quad \omega_i(t) = \cos(t); \]
otherwise,
\[ \quad v_i(t) = k \sum_{m \in N_i} x_{im}(t), \quad \omega_i(t) = \cos(t), \]
where $k$ is a small positive gain. Reference: This controller was proposed in

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Note that the controller has the form

\[ v_i = g_i(t, \{x_{im}, y_{im}\}_{m \in N_i}), \quad \omega_i = h_i(t, \{x_{im}, y_{im}\}_{m \in N_i}), \]

where \(g_i, h_i\) are smooth. Also, for every \(t_0\), \(\theta_i(t) = \theta_i(t_0) + \sin(t)\), which is periodic with period \(2\pi\). Thus the unicycles are made to constantly wiggle. The function \(g_i\) is such that

\[ \{N_i = \phi\} \Rightarrow \{v_i = 0\} \]

and

\[ \{(\forall m \in N_i) \ z_m = z_i\} \Rightarrow \{v_i = 0\}. \]

Note also that only the measurements \(x_{im}\) are needed, not \(y_{im}\). Recall that \(x_{im}\) is the projection of the relative position \(z_m - z_i\) onto the velocity vector \(r_i\). Thus this controller is allowable according to our definition.

**Running example**

\[ N_1 = \{2\}, \quad N_2 = \{3\}, \quad N_3 = \{2, 4\}, \quad N_4 = \phi \]

\[ v_1 = kx_{12} = k(z_2 - z_1)^T r_1 \]
\[ v_2 = kx_{23} = k(z_3 - z_2)^T r_2 \]
\[ v_3 = kx_{32} + kx_{34} = k(z_2 - z_3)^T r_3 + k(z_4 - z_3)^T r_3 \]
\[ v_4 = 0 \]

**Closed-loop equations**

First, assume \(N_i \neq \phi\). Then

\[ v_i = k \sum_{m \in N_i} x_{im} \]
\[ = k \sum_{m \in N_i} (z_m - z_i)^T r_i. \]

Substitute into \(\dot{z}_i = r_i v_i\) to get

\[ \dot{z}_i = k r_i \sum_{m \in N_i} (z_m - z_i)^T r_i \]
\[ = k r_i r_i^T \sum_{m \in N_i} (z_m - z_i). \]
Define the $2 \times 2$ matrix $M_i = r_i r_i^T$. Then

$$
\dot{z}_i = k M_i \sum_{m \in \mathcal{N}_i} (z_m - z_i).
$$

Second, assume $\mathcal{N}_i = \emptyset$. Then $v_i = 0$. We choose to write this as $v_i = k r_i^T 0$. Then

$$
\dot{z}_i = r_i v_i = k M_i 0.
$$

**Running example (cont’d)**

$\mathcal{N}_1 = \{2\}$, $\mathcal{N}_2 = \{3\}$, $\mathcal{N}_3 = \{2, 4\}$, $\mathcal{N}_4 = \emptyset$

$$
\begin{align*}
\dot{z}_1 &= k M_1 (z_2 - z_1) \\
\dot{z}_2 &= k M_2 (z_3 - z_2) \\
\dot{z}_3 &= k M_3 \{(z_2 - z_3) + (z_4 - z_3)\} \\
\dot{z}_4 &= k M_4 0
\end{align*}
$$

Combine:

$$
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2 \\
\dot{z}_3 \\
\dot{z}_4
\end{bmatrix} = k
\begin{bmatrix}
M_1 & 0 & 0 & 0 \\
0 & M_2 & 0 & 0 \\
0 & 0 & M_3 & 0 \\
0 & 0 & 0 & M_4
\end{bmatrix}
\begin{bmatrix}
z_2 - z_1 \\
z_3 - z_2 \\
(z_2 - z_3) + (z_4 - z_3) \\
0
\end{bmatrix}.
$$

The Laplacian of the sensor graph is

$$
L =
\begin{bmatrix}
1 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix}.
$$

Thus

$$
L \otimes I_2 =
\begin{bmatrix}
I_2 & -I_2 & 0 & 0 \\
0 & I_2 & -I_2 & 0 \\
0 & -I_2 & 2I_2 & -I_2 \\
0 & 0 & 0 & 0
\end{bmatrix}.
$$

Then

$$
-(L \otimes I_2) z =
\begin{bmatrix}
-I_2 & I_2 & 0 & 0 \\
0 & -I_2 & I_2 & 0 \\
0 & I_2 & -2I_2 & I_2 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2 \\
z_3 \\
z_4
\end{bmatrix} =
\begin{bmatrix}
z_2 - z_1 \\
z_3 - z_2 \\
(z_2 - z_3) + (z_4 - z_3) \\
0
\end{bmatrix}.
$$
Thus
\[
\frac{d}{dt} \begin{bmatrix} z_1 \\
 z_2 \\
 z_3 \\
 z_4 \\
 z \\
 \end{bmatrix} = k M \begin{bmatrix} M_1 & 0 & 0 & 0 \\
 0 & M_2 & 0 & 0 \\
 0 & 0 & M_3 & 0 \\
 0 & 0 & 0 & M_4 \\
 \end{bmatrix} \begin{bmatrix} z_2 - z_1 \\
 z_3 - z_2 \\
 (z_2 - z_3) + (z_4 - z_3) \\
 0 \\
 \end{bmatrix} - (L \otimes I_2)z.
\]

Now look at \( M_i \):
\[
M_i = r_i r_i^T,
\]
\[
M_i = \begin{bmatrix} \cos(\theta_i) \\
 \sin(\theta_i) \\
 \end{bmatrix} \begin{bmatrix} \cos(\theta_i) & \sin(\theta_i) \\
 -\sin(\theta_i) & \cos(\theta_i) \\
 \end{bmatrix} = \begin{bmatrix} \cos^2(\theta_i) & \sin(\theta_i) \cos(\theta_i) \\
 \sin(\theta_i) \cos(\theta_i) & \sin^2(\theta_i) \\
 \end{bmatrix}
\]

Since \( \theta_i(t) = \theta_i(t_0) + \sin(t) \), so \( M_i(t) \) and \( H(t) \) are \( 2\pi \)-periodic functions of \( t \).

**Summary for \( n > 1 \) unicycles**

\[
M_i(t) = \begin{bmatrix} \cos^2(\theta_i(t)) & \cos(\theta_i(t)) \sin(\theta_i(t)) \\
 \cos(\theta_i(t)) \sin(\theta_i(t)) & \sin^2(\theta_i(t)) \\
 \end{bmatrix}
\]

\[
H(t) = \text{diag}(M_1(t), \ldots, M_n(t))
\]

The closed-loop system is
\[
\dot{z}(t) = -kH(t)(L \otimes I_2)z(t)
\]

—linear, periodically time-varying system.

Convergence of the unicycles reduces to studying a periodically time-varying linear system. This is a *post facto* motivation for the control law \( \omega_i(t) = \cos(t) \). The other thing we know now is that \( L \) has a simple eigenvalue at 0, all others having positive real part.

Let us temporarily drop the subscript \( i \) in \( \theta_i \) and \( M_i \). Look at the function \( \cos^2(\theta(t)) \). Its average value over one period is
\[
\frac{1}{2\pi} \int_0^{2\pi} \cos^2(\theta(t))dt.
\]
Likewise, the average of \( M(t) \) is
\[
\overline{M} := \begin{bmatrix} m_1 & m_2 \\
 m_2 & m_3 \\
 \end{bmatrix}
\]
\[
= \begin{bmatrix} \frac{1}{2\pi} \int_0^{2\pi} \cos^2(\theta(t))dt & \frac{1}{2\pi} \int_0^{2\pi} \cos(\theta(t)) \sin(\theta(t))dt \\
 \frac{1}{2\pi} \int_0^{2\pi} \cos(\theta(t)) \sin(\theta(t))dt & \frac{1}{2\pi} \int_0^{2\pi} \sin^2(\theta(t))dt
\end{bmatrix}
\]
**Claim 2** \( \overline{M} \) is positive definite.

**Proof** A symmetric matrix is positive definite iff its principle minors are positive. Since \( m_1 > 0 \), we just have to show \( \det(M) > 0 \), i.e., \( m_1m_3 > m_2^2 \).

Define \( x(t) = \cos(\theta(t)), \ y(t) = \sin(\theta(t)) \) and the inner product
\[
\langle x, y \rangle = \frac{1}{2\pi} \int_0^{2\pi} x(t)y(t)dt.
\]
Likewise for \( \langle x, x \rangle \) and \( \langle y, y \rangle \). Then \( m_2^2 < m_1m_3 \) is equivalent to
\[
\langle x, y \rangle^2 < \langle x, x \rangle \langle y, y \rangle.
\]
The inequality \( \langle x, y \rangle^2 \leq \langle x, x \rangle \langle y, y \rangle \) is the Cauchy-Schwarz inequality and is always true. Equality holds iff \( x \) is a scalar multiple of \( y \) or vice versa. Neither is the case here—since \( \theta(t) \) is time-varying, we can’t have \( \cos(\theta(t)) = \frac{1}{c}\sin(\theta(t)) \).

\[\square\]

So \( \overline{M}_1, \ldots, \overline{M}_n \) are positive definite, and hence so is the average of \( H(t) \),
\[
\overline{H} = \text{diag}(\overline{M}_1, \ldots, \overline{M}_n).
\]
With the periodically time-varying (PTV) linear system
\[
\dot{z}(t) = -kH(t)(L \otimes I_2)z(t)
\]
we associate the time-invariant (TI) linear system
\[
\dot{z}(t) = -k\overline{H}(L \otimes I_2)z(t).
\]
We’ll prove convergence in the TI system, and then that this implies convergence in the PTV system for small enough \( k \). First we have to study \( L \otimes I_2 \), and then \( \overline{H}(L \otimes I_2) \).

**Running example** (cont’d)

\( G \) has a strongly reachable node.

\[
L = \begin{bmatrix}
1 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
simple eigenvalue at 0 with eigenvector \( 1 \); others have real part > 0

\[
L \otimes I_2 = \begin{bmatrix}
I_2 & -I_2 & 0 & 0 \\
0 & I_2 & -I_2 & 0 \\
0 & -I_2 & 2I_2 & -I_2 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
eigenvalue at 0, algebraic multiplicity 2, by KPd; geometric multiplicity 2 since rank=6; others have real part > 0 by KPd; eigenvector corresponding to 0

\[
\begin{bmatrix}
a \\
a \\
a \\
a
\end{bmatrix} = 1 \otimes a, \quad 0 \neq a \in \mathbb{R}^2
\]
Recall our closed-loop system
\[
\dot{z}(t) = -kH(t)(L \otimes I_2)z(t)
\]
and the associated (averaged) TI system
\[
\dot{z}(t) = -k\Omega(L \otimes I_2)z(t).
\]
Since the sensor graph has a globally reachable node, $-L$ has a simple eigenvalue at 0 with eigenvector $1$, all other eigenvalues having negative real part. Thus $-(L \otimes I_2)$ has an eigenvalue at 0 with algebraic and geometric multiplicity 2 and with eigenvector $1 \otimes a$, $0 \neq a \in \mathbb{R}^2$, all other eigenvalues having negative real part.

Now what about $-k\Omega(L \otimes I_2)$, where $k\Omega$ is block-diagonal with $2 \times 2$ blocks and positive definite. Certainly 0 is an eigenvalue with algebraic and geometric multiplicity 2 and with eigenvector $1 \otimes a$, $0 \neq a \in \mathbb{R}^2$. We need to prove all other eigenvalues having negative real part.

It’s convenient to introduce a term: a $2n \times 2n$ matrix $P$ has Property † if 0 is an eigenvalue with algebraic and geometric multiplicity 2 and all other eigenvalues have negative real part. So we want to prove that $-k\Omega(L \otimes I_2)$ has Property † knowing that $-(L \otimes I_2)$ does. This result is not obvious.

Time out for—

**Weighted Digraphs**

Consider a nonnegative matrix $A$. In the section on non-negative matrices we constructed a digraph $G(A)$ and proved: (1) $A$ is irreducible iff $G(A)$ is strongly connected; (2) if $G(A)$ is strongly connected, then 0 is a simple eigenvalue of $L$. Here we define a corresponding *weighted* digraph $G_w(A)$: There’s an arc from node $i$ to node $j$ iff $a_{ij} > 0$; and then the weight assigned to this arc is $a_{ij}$. The *out-degree* of a node is the sum of the weights of the arcs leaving the node. (An unweighted digraph is just a weighted digraph with all weights =1.) The weighted degree matrix $D_w$ is the diagonal matrix of out-degrees. The weighted Laplacian is then still $L_w = D_w - A$.

**Example** Let
\[
A = \begin{bmatrix}
0 & 2 & 0 \\
0 & 0 & 2.5 \\
1 & 0 & 0
\end{bmatrix}.
\]
Then the weighted digraph is

![Weighted Digraph](image)

and
\[
D_w = \begin{bmatrix}
2 & 0 & 0 \\
0 & 2.5 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad L_w = \begin{bmatrix}
2 & -2 & 0 \\
0 & 2.5 & -2.5 \\
-1 & 0 & 1
\end{bmatrix}.
\]
Observe: $A$ is irreducible; the graph is strongly connected; $L_w \mathbf{1} = 0$ and therefore 0 is an eigenvalue and $\mathbf{1}$ is a corresponding eigenvector; the other eigenvalues have positive real part by Geršgorin.

**Lemma 21** If $A$ is an irreducible non-negative matrix, then $G_w(A)$ is strongly connected, 0 is a simple eigenvalue of $L_w$ and $\mathbf{1}$ is a corresponding eigenvector, and all other eigenvalues have positive real part.

**Proof** Assume $A$ is irreducible and non-negative. The graphs $G_w(A)$ and $G(A)$ have exactly the same directed arcs. Thus $G_w(A)$ is strongly connected. And $L_w \mathbf{1} = 0$ by construction. We have

$$L_w = D_w - A = D_w (I - D_w^{-1} A).$$

Define

$$\tilde{A} = D_w^{-1} A, \quad \tilde{L}_w = D_w^{-1} L_w = I - \tilde{A}.$$ 

Notice that $\tilde{A} \geq 0$ and the two matrices $A, \tilde{A}$ have zeros in exactly the same locations. Thus $\tilde{A}$ is irreducible too. Then by Frobenius’ theorem, $\rho(\tilde{A})$ is a simple eigenvalue of $\tilde{A}$. Since $\tilde{A}$ has constant row sums, namely 1, in fact $\rho(\tilde{A}) = \|\tilde{A}\|_{\infty} = 1$. It follows that 0 is a simple eigenvalue of $\tilde{L}_w = I - \tilde{A}$. Therefore, $\tilde{L}_w$ has rank $n - 1$, hence so does $L_w = D_w \tilde{L}_w$. Thus 0 is a simple eigenvalue of $L_w$.

Geršgorin’s theorem implies that the nonzero eigenvalues have positive real part. $\square$

**Lemma 22** If the sensor digraph is strongly connected, then $-k\overline{H}(L \otimes I_2)$ has Property $\dagger$.

**Proof** Assume the sensor digraph is strongly connected. Let $A$ be its adjacency matrix and $D$ its degree matrix; so $L = D - A$. We have $L \mathbf{1} = 0$. We’ll now show that $L$ has a positive left-eigenvector corresponding to the eigenvalue 0; that is, $p^T L = 0$, $p > 0$.

As usual, define $\tilde{L} := D^{-1} L$ and $\tilde{A} := D^{-1} A$ to get $\tilde{L} = I - \tilde{A}$. The matrices $A$ and $\tilde{A}$ have the same distribution of zero entries and nonzero entries, and so the associated digraph $G(\tilde{A})$ is also strongly connected and therefore $\tilde{A}$ is irreducible. In addition, the row sums of $\tilde{A}$ are all 0, so it follows that $\rho(\tilde{A}) = 1$. Hence, $\rho(\tilde{A}^T) = 1$. By Frobenius’ theorem $\tilde{A}^T$ has a corresponding positive eigenvector: $\tilde{A}^T \tilde{p} = \tilde{p}$, or $\tilde{p}^T \tilde{A} = \tilde{p}^T$. Then $\tilde{p}^T \tilde{L} = 0$, and so $p^T L = 0$, where $p^T = \tilde{p}^T D^{-1}$.

Now define

$$P = \text{diag}(p_1, p_2, \ldots, p_n), \quad Q = L^T P + P L,$$

two symmetric matrices. Then

$$Q \mathbf{1} = L^T P \mathbf{1} + P L \mathbf{1} = L^T P \mathbf{1} = L^T \mathbf{p} = 0.$$ 

So 0 is an eigenvalue of $Q$ and $\mathbf{1}$ is an eigenvector. Also,

$$Q = L^T P + P L = (D - A)^T P + P (D - A) = (D^T P + PD) - (A^T P + PA), \quad (5.6)$$

where $D^T P + PD$ is a positive diagonal matrix. Now $A$ is nonnegative irreducible (the digraph is strongly connected), hence so is $PA$, hence so is $A^T P + PA$ (irreducible + nonnegative = irreducible).
Now consider the weighted digraph $G_w$ with adjacency matrix $A^T P + PA$. The weighted degree matrix is $D^T P + PD$; this is because

$$(D^T P + PD)\mathbf{1} = (A^T P + PA)\mathbf{1}.$$\

Also, from (5.6) the weighted Laplacian is $Q$. From Lemma 21, $G_w$ is strongly connected, 0 is a simple eigenvalue of $Q$, and the others have positive real part. Thus $Q$ is positive semi-definite.

We now derive a Lyapunov equation for $L \otimes I_2$:

$$(L \otimes I_2)^T(P \otimes I_2) + (P \otimes I_2)(L \otimes I_2) = (L^T \otimes I_2)(P \otimes I_2) + (P \otimes I_2)(L \otimes I_2) \quad \text{by KPb}$$

$$= (L^T P) \otimes I_2 + (PL) \otimes I_2 \quad \text{by KPc}$$

$$= (L^T P + PL) \otimes I_2 \quad \text{by KPa}$$

$$= Q \otimes I_2.$$\

To simplify notation, define

$$L_2 = L \otimes I_2, \quad P_2 = P \otimes I_2, \quad Q_2 = Q \otimes I_2.$$\

Then

$$L_2^T P_2 + P_2 L_2 = Q_2,$$\

a Lyapunov equation. By KPd, 0 is an eigenvalue of $L_2$ of algebraic multiplicity 2, and the other eigenvalues have positive real part; likewise for $Q_2$. Therefore $Q_2$ is positive semi-definite. In addition, $\ker(Q) = \ker(L)$, since both are 1-dimensional and contain the vector $\mathbf{1}$, and therefore $\ker(Q_2) = \ker(L_2)$.

Now set $R = k\mathbf{II}$, a block-diagonal positive-definite matrix with $2 \times 2$ blocks. From the Lyapunov equation,

$$L_2^T RR^{-1} P_2 + P_2 R^{-1} RL_2 = Q_2,$$\

i.e.,

$$(RL_2)^T(R^{-1} P_2) + (P_2 R^{-1})(RL_2) = Q_2. \quad (5.7)$$\

Now, by their structures, $R^{-1}$ and $P_2$ commute: They’re block-diagonal with $2 \times 2$ blocks and the blocks of $P_2$ are scalar blocks—they have the form $p_i I_2$. Thus $R^{-1} P_2$ is symmetric and the preceding equation is a Lyapunov equation.

Since $R^{-1} P_2$ is positive definite and $Q_2$ positive semi-definite, by (5.7) the eigenvalues of $RL_2$ have real parts $\geq 0$. But we already saw that 0 is an eigenvalue of $L_2$ of algebraic multiplicity 2, so likewise for $RL_2$. Therefore to show that the other eigenvalues have positive real part, we only need to show that no other eigenvalues of $RL_2$ except the 2 zero eigenvalues are on the imaginary axis.

Suppose that $RL_2 x = j\omega x$ with $x \neq 0$. Then

$$0 = x^* (RL_2)^T(R^{-1} P_2) + (P_2 R^{-1})(RL_2) - Q_2 \ x$$

$$= -j\omega x^*(R^{-1} P_2) x + j\omega x^*(P_2 R^{-1}) x - x^* Q_2 x$$

$$= -x^* Q_2 x.$$\

It follows that $x \in \ker(Q_2) = \ker(L_2)$ and therefore $\omega = 0$. \hfill \Box

To strengthen the preceding lemma to the case where the sensor digraph just has a globally reachable node, we need a specialized result about Lyapunov equations from
Lemma 23 Suppose $A$ is a non-negative matrix with $\rho(A) < 1$. There exists a diagonal positive definite matrix $P$ such that $(I - A)^T P + P(I - A)$ is positive definite.

The proof of this result would take us too far afield, into $M$-matrices, so we omit it.

Lemma 24 If the sensor digraph has a globally reachable node, then $-k\Pi(L \otimes I_2)$ has Property $\dagger$.

Proof Let $\mathcal{V}'$ be the set of all globally reachable nodes. Say it has $r$ elements.

If $r = n$, the result is from Lemma 22. So assume $1 \leq r < n$. By re-ordering, we may suppose $\mathcal{V}'$ consists of nodes $1, \ldots, r$. Then the adjacency and degree matrices have the forms

$$A = \begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix}, \quad D = \begin{bmatrix} D_1 & 0 \\ 0 & D_3 \end{bmatrix},$$

so the Laplacian is

$$L = \begin{bmatrix} L_1 & 0 \\ L_2 & L_3 \end{bmatrix} = \begin{bmatrix} D_1 & 0 \\ 0 & D_3 \end{bmatrix} - \begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix}.$$

Note that the digraph $\mathcal{G}(A_1)$ is strongly connected.

Running example The only strongly reachable node is 4. Re-order the nodes 4, 1, 2, 3. Then

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}.$$

Note that $L_3$ is nonsingular. (We proved in the unit on point robots, time-invariant sensor graph that $L_3$ is nonsingular in general.) Note also that $\rho(D_3^{-1}A_3) < 1$:

$$D_3^{-1}A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0.5 & 0 \end{bmatrix}.$$

Let $R = k\Pi$ and partition it as

$$R = \begin{bmatrix} R_1 & 0 \\ 0 & R_3 \end{bmatrix}.$$

We have

$$L \otimes I_2 = \begin{bmatrix} L_1 \otimes I_2 & 0 \\ L_2 \otimes I_2 & L_3 \otimes I_2 \end{bmatrix}.$$
and therefore
\[ R(L \otimes I_2) = \begin{bmatrix} R_1(L_1 \otimes I_2) & 0 \\ R_3(L_2 \otimes I_2) & R_3(L_3 \otimes I_2) \end{bmatrix}. \]

By Lemma 22, \(-R_1(L_1 \otimes I_2)\) has Property †. It therefore remains to show that all the eigenvalues of \(R_3(L_3 \otimes I_2)\) have positive real part. We have
\[ L_3 = D_3 - A_3 = D_3(I - D_3^{-1}A_3) =: D_3(I - \tilde{A}_3) \]
and \(\rho(\tilde{A}_3) < 1\). By Lemma 23, there exists a diagonal positive definite matrix \(\tilde{P}\) such that
\[ (I - \tilde{A}_3)^T \tilde{P} + \tilde{P}(I - \tilde{A}_3) \]
is positive definite. Define \(P = \tilde{P}D_3^{-1}\). Then
\[ Q := L_3^T P + PL_3 \]
is positive definite. Applying the Kronecker product with \(I_2\) to both sides of the above equation and simplifying yields
\[ Q \otimes I_2 = (L_3 \otimes I_2)^T (P \otimes I_2) + (P \otimes I_2)(L_3 \otimes I_2). \]
Set \(\tilde{P} = (P \otimes I_2)R_3^{-1} = R_3^{-1}(P \otimes I_2)\) to get
\[ (R_2(L_3 \otimes I_2))^T \tilde{P} + \tilde{P}(R_2(L_3 \otimes I_2)) = Q \otimes I_2. \]
Since \(\tilde{P}\) and \(Q \otimes I_2\) are positive definite, the eigenvalues of \(R_3(L_3 \otimes I_2)\) have positive real part. □

Recap The \(n > 1\) unicycles are controlled via the equations
\[ \omega_i(t) = \cos(t), \quad v_i(t) = \begin{cases} 0, & N_i = \phi \\ k\sum_{m \in N_i} (z_m - z_i)^T r_i, & N_i \neq \phi. \end{cases} \]
The controlled system is linear periodically time-varying,
\[ \dot{z}(t) = -kH(t)(L \otimes I_2)z(t), \quad (5.8) \]
where \(L\) is the Laplacian of the sensor digraph and
\[ M_i(t) = r_i(t)r_i(t)^T, \quad H(t) = \text{diag}(M_1(t), \ldots, M_n(t)). \]
The associated (averaged) time-invariant system is
\[ \dot{z}(t) = -k\overline{H}(L \otimes I_2)z(t), \quad (5.9) \]
where \(\overline{H}\) is the average of \(H(t)\) over one period. Under the assumption that the sensor digraph has a globally reachable node, we’ve proved that \(-(L \otimes I_2)\) and \(-\overline{H}(L \otimes I_2)\) have Property †.

Running example (cont’d)
\[ L = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

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Do an eigenanalysis—matrix of eigenvectors:

\[
F = \begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Then

\[
LF = \tilde{F}L,
\]

\[
\tilde{L} = \begin{bmatrix}
L_{11} & 0 \\
0 & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 2 & -1 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

So we get in turn, using KPc,

\[
(LF) \otimes (I_2) = (F \tilde{L}) \otimes (I_2)
\]

\[
(LF) \otimes (I_2 I_2) = (F \tilde{L}) \otimes (I_2 I_2)
\]

\[
(L \otimes I_2)(F \otimes I_2) = (F \otimes I_2)(\tilde{L} \otimes I_2)
\]

\[
(F \otimes I_2)^{-1}(L \otimes I_2)(F \otimes I_2) = \tilde{L} \otimes I_2 = \begin{bmatrix}
I_2 & 0 & 0 & 0 \\
0 & 2I_2 & -I_2 & 0 \\
0 & -I_2 & I_2 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
F \otimes I_2 = \begin{bmatrix}
I_2 & I_2 & I_2 & I_2 \\
0 & 0 & I_2 & I_2 \\
0 & I_2 & 0 & I_2 \\
0 & 0 & 0 & I_2
\end{bmatrix}
\]

There is a similarity transformation \( F \) such that

\[
-F^{-1}LF = \begin{bmatrix}
-L_{11} & 0 \\
0 & 0
\end{bmatrix},
\]

where \(-L_{11}\) is stable and the last column of \( F \) is \( 1 \). Then

\[
(F \otimes I_2)^{-1}(L \otimes I_2)(F \otimes I_2) = \begin{bmatrix}
-(L_{11} \otimes I_2) & 0 \\
0 & 0
\end{bmatrix},
\]

where \(-(L_{11} \otimes I_2)\) is stable and the last two columns of \( F \otimes I_2 \) are

\[
1 \otimes I_2 = \begin{bmatrix}
I_2 \\
I_2 \\
I_2 \\
I_2
\end{bmatrix}.
\]
Apply the transformation \( z = (F \otimes I_2)e \) to
\[
\dot{z} = -kH(t)(L \otimes I_2)z
\]
to get
\[
\dot{e} = -k(F \otimes I_2)^{-1}H(t)(L \otimes I_2)(F \otimes I_2)e
\]
\[
= -k \{(F \otimes I_2)^{-1}H(t)(F \otimes I_2)\} \{(F \otimes I_2)^{-1}(L \otimes I_2)(F \otimes I_2)\} e.
\]
Partition \( e \) as \( e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \), where \( \dim e_2 = 2 \). Then we have
\[
\begin{bmatrix}
\dot{e}_1 \\
\dot{e}_2
\end{bmatrix} = -k \{(F \otimes I_2)^{-1}H(t)(F \otimes I_2)\} \begin{bmatrix}
-(L_{11} \otimes I_2) & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
e_1 \\
e_2
\end{bmatrix}.
\]
This has the form
\[
\begin{bmatrix}
\dot{e}_1 \\
\dot{e}_2
\end{bmatrix} = k \begin{bmatrix}
A_{11}(t) & 0 \\
A_{21}(t) & 0
\end{bmatrix} \begin{bmatrix}
e_1 \\
e_2
\end{bmatrix},
\]
i.e.,
\[
\dot{e}_1 = kA_{11}(t)e_1, \quad \dot{e}_2 = kA_{21}(t)e_1.
\]
Notice that \( A_{11}(t), A_{21}(t) \) are periodic.
Likewise, apply the transformation \( z = (F \otimes I_2)e \) to the averaged system
\[
\dot{z} = -k\bar{H}(L \otimes I_2)z
\]
to get
\[
\dot{e} = -k(F \otimes I_2)^{-1}\bar{H}(L \otimes I_2)(F \otimes I_2)e
\]
\[
= -k \{(F \otimes I_2)^{-1}\bar{H}(F \otimes I_2)\} \{(F \otimes I_2)^{-1}(L \otimes I_2)(F \otimes I_2)\} e.
\]
Then we have
\[
\begin{bmatrix}
\dot{e}_1 \\
\dot{e}_2
\end{bmatrix} = -k \{(F \otimes I_2)^{-1}\bar{H}(F \otimes I_2)\} \begin{bmatrix}
-(L_{11} \otimes I_2) & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
e_1 \\
e_2
\end{bmatrix}.
\]
This has the form
\[
\begin{bmatrix}
\dot{e}_1 \\
\dot{e}_2
\end{bmatrix} = k \begin{bmatrix}
\bar{A}_{11} & 0 \\
\bar{A}_{21} & 0
\end{bmatrix} \begin{bmatrix}
e_1 \\
e_2
\end{bmatrix},
\]
i.e.,
\[
\dot{e}_1 = k\bar{A}_{11}e_1, \quad \dot{e}_2 = k\bar{A}_{21}e_1.
\]
As we showed before, \( \bar{A}_{11} \) is stable. Hence as \( t \to \infty \), \( e_1(t) \to 0 \) and \( e_2(t) \) converges to a constant \( 2 - \dim \) vector, say \( z_{ss} \). Thus for this averaged system and \( \forall k > 0 \)
\[
\lim_{t \to \infty} z(t) = (F \otimes I_2) \begin{bmatrix}
0 \\
z_{ss}
\end{bmatrix} = (1 \otimes I_2)z_{ss} = 1 \otimes z_{ss}.
\]
Now let us return to the periodically time-varying system
\[
\dot{e}_1 = kA_{11}(t)e_1, \quad \dot{e}_2 = kA_{21}(t)e_1.
\]
Suppose we can show that, for small enough \( k > 0 \), \( e_1(t) \) converges to 0 exponentially for every initial condition. It will follow, since \( A_{21}(t) \) is periodic and hence bounded, that \( e_2(t) \) converges to a constant vector for every initial condition. Then we’ll be done. So what remains is the following averaging theory.
The System $\dot{x} = A(t)x$

We begin by reviewing the general linear time-varying system

$$
\dot{x}(t) = A(t)x(t).
$$

(5.10)

The transition matrix of (5.10) is the matrix that maps the state at one time, say $t_0$, to the state at another time, say $t$:

$$
x(t) = \Phi(t, t_0)x(t_0).
$$

In general, there's no closed-form expression for $\Phi(t, t_0)$ in terms of $A(t)$ except in some special cases.

1. As you well know, if $A(t) = A$, a constant matrix, then

$$
\Phi(t, t_0) = e^{A(t-t_0)}.
$$

2. If $A(t)$ is a scalar ($1 \times 1$ matrix), then

$$
\Phi(t, t_0) = e^{\int_{t_0}^{t} A(\tau)d\tau}.
$$

3. If, for every value of $t_1$ and $t_2$, $A(t)$ and $\int_{t_1}^{t_2} A(\tau)d\tau$ commute, then

$$
\Phi(t, t_0) = e^{\int_{t_0}^{t} A(\tau)d\tau}.
$$

**Theorem 34** Let $A(t)$ be periodic of period $T$. Suppose that

$$
\bar{A} = \frac{1}{T} \int_{0}^{T} A(\sigma)d\sigma
$$

has all its eigenvalues in the half plane $\Re(s) < 0$. Then there exists $\varepsilon_0 > 0$ such that the origin of

$$
\dot{x}(t) = \varepsilon A(t)x(t)
$$

is exponentially stable for all $0 < \varepsilon < \varepsilon_0$.

**Proof for $n = 1$** The case $n = 1$ is particularly easy. In that case, let’s write $a(t)$ for $A(t)$. The solution of $\dot{x}(t) = a(t)x(t)$ is

$$
x(t) = e^{\int_{t_0}^{t} a(\tau)d\tau}x(t_0).
$$

Each $t > t_0$ lies in one of the intervals

$$
[t_0, t_0 + T), [t_0 + T, t_0 + 2T), \ldots
$$

Say $t_0 + mT \leq t < t_0 + (m + 1)T$. Writing $\Delta t = t - (t_0 + mT)$, we have

$$
x(t) = e^{\int_{t_0}^{t_0 + mT} a(\tau)d\tau}x(t_0)
\quad = e^{\int_{t_0}^{t_0 + mT} a(\tau)d\tau + \int_{t_0 + mT}^{t} a(\tau + \Delta t)d\tau}x(t_0)
\quad = e^{mT\bar{a} + \int_{t_0 + mT}^{t} a(\tau + \Delta t)d\tau}x(t_0)
\quad = e^{mT\bar{a} + \int_{t_0}^{t_0 + mT} a(\tau + \Delta t)d\tau}x(t_0).
$$

Now $e^{\int_{t_0}^{t} a(\tau + \Delta t)d\tau}$ is a bounded function for $0 \leq \Delta t < T$, and $e^{mT\bar{a}}$ converges exponentially to 0 as $m \to \infty$, since $\bar{a} < 0$. Thus $x(t)$ converges exponentially to 0 as $t \to \infty$.

Proof for general $n$: Omitted: See Khalil if you like.
5.4.4 Forming a Line

Problem 2 Find, if possible, an allowable controller such that $(\forall t_0)(\forall x_i(t_0), y_i(t_0), \theta_i(t_0); i = 1, \ldots, n)$ all vehicles converge to form a line.

Theorem 35 Problem 2 is solvable if and only if $G$ has at most two disjoint closed sets of nodes.

Proof of ($\Rightarrow$) Fix any allowable controller. Suppose $G$ has three nonempty disjoint closed sets of nodes—say, $N_1, N_2, N_3$. Start all the unicycles in $N_1$ at some initial condition $z_i(0) = a_1$, all the unicycles in $N_2$ at some initial condition $z_i(0) = a_2$, and all the unicycles in $N_3$ at some initial condition $z_i(0) = a_3$. These three groups of unicycles will not move in the plane (though they may rotate), and hence, if $a_1, a_2, a_3$ are not collinear, convergence to a line does not occur.

Proof of ($\Leftarrow$) Suppose $G$ has at most two disjoint closed sets of nodes. In view of the earlier lemma

Assume $G$ has at least 2 nodes. It has a globally reachable node iff it does not have two disjoint closed subsets of nodes.

Running example

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
\end{array}
\]

$N_1 = \{1, 2\}, \ N_2 = \{3, 4\}$

Back to proof: The induced subgraph $G_1 = (N_1, E \cap (N_1 \times N_1))$ has a globally reachable node. For if not, then it has two disjoint closed sets of nodes; together with $N_2$, these make three disjoint closed sets in $G$, a contradiction.

Running example (cont’d)

\[
\begin{array}{ccc}
G_1 & 1 & 2 \\
\end{array}
\]

Back to proof: Likewise, $G_2 = (N_2, E \cap (N_2 \times N_2))$ has a globally reachable node.

Case 1 $N_1 \cup N_2 = \mathcal{V}$

By Theorem 33 all unicycles in $N_1$ can converge to a point, and likewise for all unicycles in $N_2$. Two points form a line (they’re collinear).

Case 2 $N_1 \cup N_2 \neq \mathcal{V}$
Let $\mathcal{N}_3$ be the complement of $\mathcal{N}_1 \cup \mathcal{N}_2$ in $\mathcal{V}$. Every node in $\mathcal{N}_3$ can reach either $\mathcal{N}_1$ or $\mathcal{N}_2$. If necessary, re-order the nodes as $\mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3$.

Then the Laplacian has the form

$$L = \begin{bmatrix}
L_{11} & 0 & 0 \\
0 & L_{22} & 0 \\
L_{31} & L_{32} & L_{33}
\end{bmatrix}.$$ 

**Running example (cont'd)**

\[A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0
\end{bmatrix}, L = \begin{bmatrix}
1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 \\
0 & 0 & -1 & 1 & 0 \\
-1 & -1 & 0 & -1 & 3
\end{bmatrix}
\]

Notice that $L$ has two zero eigenvalues and one is in $L_{11}$ and the other is in $L_{22}$. The matrix $-L_{33}$ is stable.

Back to proof: Now apply the controller of before: If $\mathcal{N}_i = \emptyset$, then

$$v_i = 0, \quad \omega_i(t) = \cos(t);$$

otherwise,

$$v_i = k \sum_{m \in \mathcal{N}_i} (\mathbf{z}_m - \mathbf{z}_i)^T \mathbf{r}_i, \quad \omega_i(t) = \cos(t),$$

where $k$ is a small positive gain.

**Running example (cont'd)**

\[
\begin{align*}
\dot{\mathbf{z}}_1 &= kM_1(\mathbf{z}_2 - \mathbf{z}_1) \\
\dot{\mathbf{z}}_2 &= kM_2 \mathbf{0} \\
\dot{\mathbf{z}}_3 &= kM_3(\mathbf{z}_4 - \mathbf{z}_3) \\
\dot{\mathbf{z}}_4 &= kM_4(\mathbf{z}_3 - \mathbf{z}_4) \\
\dot{\mathbf{z}}_5 &= kM_5\{(\mathbf{z}_1 - \mathbf{z}_5) + (\mathbf{z}_2 - \mathbf{z}_5) + (\mathbf{z}_4 - \mathbf{z}_5)\}
\end{align*}
\]
I.e.,

$$\dot{z} = -kH(L \otimes I_2)z$$

Corresponding to the partition

$$L = \begin{bmatrix} L_{11} & 0 & 0 \\ 0 & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix},$$

partition $z$ as

$$z = \begin{bmatrix} z^1 \\ z^2 \\ z^3 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix}$$
in this example.

Then

$$\dot{z}^1 = -kH_1(t)(L_{11} \otimes I_2)z^1,$$
$$\dot{z}^2 = -kH_2(t)(L_{22} \otimes I_2)z^2,$$
$$\dot{z}^3 = -kH_3(t)\left\{(L_{31} \otimes I_2)z^1 + (L_{32} \otimes I_2)z^2 + (L_{33} \otimes I_2)z^3\right\},$$

where, in this example,

$$H_1 = \text{diag}(M_1, M_2), H_2 = \text{diag}(M_3, M_4), H_3 = M_5.$$ 

Back to proof: From Theorem 33, for sufficiently small $k$, there exist $z^1_{ss}, z^2_{ss} \in \mathbb{R}^2$ such that

$$\lim_{t \to \infty} z^1(t) = 1 \otimes z^1_{ss}, \quad \lim_{t \to \infty} z^2(t) = 1 \otimes z^2_{ss}$$

with exponential convergence rate. Define $\zeta$ via

$$\dot{z}^3 = -kH_3(t)\left\{(L_{31} \otimes I_2)z^1 + (L_{32} \otimes I_2)z^2 + (L_{33} \otimes I_2)z^3\right\},$$

Then

$$\dot{\zeta} = (L_{31} \otimes I_2)\dot{z}^1 + (L_{32} \otimes I_2)\dot{z}^2 + (L_{33} \otimes I_2)\dot{z}^3$$
$$= -k(L_{31} \otimes I_2)H_1(t)(L_{11} \otimes I_2)z^1$$
$$-k(L_{32} \otimes I_2)H_2(t)(L_{22} \otimes I_2)z^2$$
$$-k(L_{33} \otimes I_2)H_3(t)\zeta,$$

which has the form

$$\dot{\zeta} = -k(L_{33} \otimes I_2)H_3(t)\zeta + v.$$
Since $L_{11}1 = 0$, so
\[(L_{11} \otimes I_2)(1 \otimes z_{ss}^1) = 0;\]
likewise
\[(L_{22} \otimes I_2)(1 \otimes z_{ss}^2) = 0.\]
Thus $v$ converges to 0 exponentially.

Since $-L_{33}$ is stable, so is $-L_{33}^T$. By an argument similar to the one in the proof of Lemma 24, 
\[-(L_{33} \otimes I_2)H_3\) is stable too. Thus the origin of
\[\dot{\zeta} = -k(L_{33} \otimes I_2)H_3\zeta\]
is globally exponentially stable. Hence by averaging theory, for small enough $k$ the origin of the system
\[\dot{\zeta} = -k(L_{33} \otimes I_2)H_3(t)\zeta\]
is globally exponentially stable. Hence
\[\dot{\zeta} = -k(L_{33} \otimes I_2)H_3(t)\zeta + v.\]
can be viewed as an exponentially stable system with an exponentially vanishing input. It turns out
that this implies $\zeta$ converges exponentially to 0, again, for small enough $k$. Thus, for small enough $k$,
\[
\lim_{t \to \infty} \{(L_{31} \otimes I_2)z^1(t) + (L_{32} \otimes I_2)z^2(t) + (L_{33} \otimes I_2)z^3(t)\} = 0
\]
\[
\Rightarrow
\]
\[(L_{31} \otimes I_2)(1 \otimes z_{ss}^1) + (L_{32} \otimes I_2)(1 \otimes z_{ss}^2) + (L_{33} \otimes I_2)z^3(\infty) = 0
\]
\[
\Rightarrow
\]
\[(L_{31}1) \otimes z_{ss}^1 + (L_{32}1) \otimes z_{ss}^2 + (L_{33} \otimes I_2)z^3(\infty) = 0
\]
by KPd.

**Running example (cont’d)**

\[
L = \begin{bmatrix}
1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 \\
0 & 0 & -1 & 1 & 0 \\
-1 & -1 & 0 & -1 & 3
\end{bmatrix}
\]

From
\[(L_{31}1) \otimes z_{ss}^1 + (L_{32}1) \otimes z_{ss}^2 + (L_{33} \otimes I_2)z^3(\infty) = 0\]

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we have
\[-2z^{1}_{ss} - 1z^{2}_{ss} + 3z^{3}(\infty) = 0\]
i.e.,
\[z^{3}(\infty) = \frac{2}{3}z^{1}_{ss} + \frac{1}{3}z^{2}_{ss}.\]
Thus $z^{3}(\infty)$ is a convex combination of $z^{1}_{ss}, z^{2}_{ss}$:

![Diagram showing convex combination of vectors $z^{1}_{ss}, z^{2}_{ss}$ and $z^{3}(\infty)$]

Back to proof: $L1 = 0$ and hence
\[
\begin{bmatrix}
L_{31} & L_{32} & L_{33}
\end{bmatrix}
1 = 0
\Rightarrow
\begin{bmatrix}
L_{33}^{-1}L_{31} & L_{33}^{-1}L_{32} & I
\end{bmatrix}
1 = 0
\Rightarrow
L_{33}^{-1}L_{31}1 + L_{33}^{-1}L_{32}1 + 1 = 0.
\]
Also,
\[
(L_{31} \otimes I_{2})(1 \otimes z^{1}_{ss}) + (L_{32} \otimes I_{2})(1 \otimes z^{2}_{ss}) + (L_{33} \otimes I_{2})z^{3}(\infty) = 0
\Rightarrow
(L_{33} \otimes I_{2})^{-1}(L_{31} \otimes I_{2})(1 \otimes z^{1}_{ss}) + (L_{33} \otimes I_{2})^{-1}(L_{32} \otimes I_{2})(1 \otimes z^{2}_{ss}) + z^{3}(\infty) = 0
\Rightarrow
((L_{33}^{-1}L_{31}) \otimes I_{2}) (1 \otimes z^{1}_{ss}) + ((L_{33}^{-1}L_{32}) \otimes I_{2}) (1 \otimes z^{2}_{ss}) + z^{3}(\infty) = 0
\Rightarrow
(L_{33}^{-1}L_{31}1) \otimes z^{1}_{ss} + (L_{33}^{-1}L_{32}1) \otimes z^{2}_{ss} + z^{3}(\infty) = 0.
\]
Now the vector $z^{3}(\infty)$ is made up of 2-vectors. For example, the first two components of $z^{3}(\infty)$ are the $x$ and $y$ components of the first unicycle in the node set $V_{3}$. Denote this subvector by $z^{3}_{1}(\infty)$.
It’s left for you to show that the two equations
\[
L_{33}^{-1}L_{31}1 + L_{33}^{-1}L_{32}1 + 1 = 0
\]
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imply that $z^{3}_{1}(\infty)$ is a convex combination of $z^{1}_{ss}$ and $z^{2}_{ss}$. □
5.4.5 Conclusion

The trouble with the controller is that the unicycles keep wiggling. Also, a serious limitation is that the sensor graph is time-invariant.

5.4.6 Exercises

1. Give an example of matrices $Q$, $A$ with the properties
   
   (a) $Q$ is diagonal and positive definite
   
   (b) $A$ is stable (all eigenvalues have negative real part)
   
   (c) $QA$ is not stable.

2. Consider the lemma

   *Suppose $A$ is a non-negative matrix with $\rho(A) < 1$. There exists a diagonal positive definite matrix $P$ such that $(I - A)^T P + P(I - A)$ is positive definite.*

   To prove it, one might try $P = I$. Then

   \[(I - A)^T P + P(I - A) = 2I - (A^T + A),\]

   which is positive definite iff $\rho(A^T + A) < 2$. So the choice $P = I$ would work if

   \[\rho(A) < 1 \implies \rho(A^T + A) < 2.\]

   Give a counterexample to this.

5.4.7 References

Chapter 6

Extra Topics

6.1 Rigidity Theory

Scenario: There are $n$ sensors lying randomly in some area. Some of them have GPS receivers, and hence know their global coordinates (to some accuracy). These sensors are called anchors. Because GPS devices are expensive, the remaining sensors have only proximity devices that can compute distances to nearby neighbours. This setup generates a kind of graph in the plane: The nodes are located where the sensors are, and there’s an edge between two nodes iff the two sensors can determine the distance between them. In this way, the global coordinates of some nodes are known, and also the lengths of the edges are known. When does this data uniquely determine the global coordinates of all the nodes? The answer is, when the graph is globally rigid.

This topic will likely also be relevant for studying moving formations.

6.1.1 Introduction

Consider $n$ point robots moving in the plane. Suppose they should move in an equilibrium formation in the sense that every point robot should keep a constant distance from every other point robot. This leads to the concept of a rigid formation.
In general, how many distances need to be maintained for the whole formation to be in equilibrium? Note that this is not necessarily the same as the condition that all other agents should appear to be stationary. Example: Two unicycles, A and B. A is moving at constant speed in a constant direction; B is moving in a circular motion with A as the centre of the circle. Then the distance-graph is rigid, but the vehicles are not in equilibrium in our sense.

This section is an introduction to the interesting subject of rigidity theory. It’s based on Roth’s article. Most proofs are omitted.

**Example 1** Consider the triangle in $\mathbb{R}^2$.

Suppose the coordinates are

$$p_1 = (1, 1), \quad p_2 = (1, 0), \quad p_3 = (0, 0).$$

Imagine the triangle represents a mechanical linkage: three bars of fixed lengths, together with three fully rotatable joints. This triangle is obviously rigid—let’s prove it.

Let’s fix $p_2, p_3$ and consider linkages where the first vertex is a variable $x_1$ (a vector). What values can $x_1$ take given that the bars are of fixed length? Obviously, only two: $p_1$ and its reflection $(1, -1)$. More formally, the **edge equations** for $x_1$ are

$$\|x_1 - p_2\| = \|p_1 - p_2\|, \quad \|x_1 - p_3\| = \|p_1 - p_3\|.$$

Introduce the **edge function**

$$f(x_1) = (\|x_1 - p_2\|^2, \|x_1 - p_3\|^2); \quad f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2.$$
So $x_1$ must satisfy the constraint $f(x_1) = f(p_1)$.

**Aside** Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a function from one set to another. Let $y \in \mathcal{Y}$. Then

$$f^{-1}(y) := \{x : f(x) = y\}.$$  

Notice that $f^{-1}(y)$ is a set, a **level set**, and there is no implication in this notation that $f$ has an inverse. As an example, if $\mathcal{X}$ and $\mathcal{Y}$ are vector spaces and $f$ is linear, namely, $f(x) = Ax$, $A$ a matrix, then $f^{-1}(0)$ equals the kernel of $A$.

Back to the triangle ... The vertex $x_1$ must satisfy $x_1 \in f^{-1}(f(p_1))$.

Recall the **Inverse Function Theorem**: Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuously differentiable and its Jacobian, $\partial f/\partial x$, is invertible at a point $p$. Then $f$ has a continuously differentiable inverse in a neighbourhood of $p$. Therefore, $f$ is one-to-one in a neighbourhood of $p$. That is, there is a neighbourhood $U$ of $p$ such that

$x \in U, f(x) = f(p) \implies x = p,$

i.e., $f^{-1}(f(p)) \cap U = \{p\}$.

Back to the triangle ... We had

$$f(x_1) = (\|x_1 - p_2\|^2, \|x_1 - p_3\|^2); \quad f : \mathbb{R}^2 \rightarrow \mathbb{R}^2.$$  

The Jacobian is

$$\partial f/\partial x_1 = 2 \begin{bmatrix} (x_1 - p_2)^T \\ (x_1 - p_3)^T \end{bmatrix}.$$  

Thus

$$\partial f/\partial x_1(p_1) = 2 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$$

which has rank 2.

**Conclusion** $x_1 = p_1$ is the only solution of the edge equations near $p_1$. Or, there is no other linkage with vertex near $p_1$.

---

**Example 2** Consider the square

![Diagram of a square with vertices at $p_1$, $p_2$, $p_3$, and $p_4$.]
Suppose the coordinates are

\[ p_1 = (0, 1), \ p_2 = (1, 1), \ p_3 = (1, 0), \ p_4 = (0, 0). \]

Let’s fix \( p_3, p_4 \) and consider linkages where the first and second vertices are variables \( x_1, x_2 \). Define

\[ x = (x_1, x_2) \in \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4 \]

and the edge function

\[ f(x) = (\|x_1 - p_4\|^2, \|x_1 - x_2\|^2, \|x_2 - p_3\|^2); \quad f : \mathbb{R}^4 \longrightarrow \mathbb{R}^3. \]

So \( x \) must satisfy the constraint \( f(x) = f(p), p = (p_1, p_2) \).

We need to recall the **Implicit Function Theorem**. First we’ll do it for a linear function. Suppose \( f : \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^m \) is a linear function, that is,

\[ f(x) = \begin{bmatrix} A & B \end{bmatrix} x, \]

where \( A \) is an \( m \times n \) matrix and \( B \) is an \( m \times m \) matrix. Suppose \( B \) is invertible. Defining \( G = -B^{-1}A \), we have that the kernel of \( f \) is

\[ \left\{ \begin{bmatrix} I \\ G \end{bmatrix} v : v \in \mathbb{R}^n \right\}. \]

That is, every solution of the equation \( f(a, b) = 0 \) has the form \( b = Ga \).

More generally but for the same function: Let \( (a, b) \) be an arbitrary vector in \( \mathbb{R}^{n+m} \) and define \( c = f(a, b) \). Then the general solution of \( c = f(x) \) is

\[ x = \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} I \\ G \end{bmatrix} v, \quad v \in \mathbb{R}^n. \]

Defining the function \( g : \mathbb{R}^n \longrightarrow \mathbb{R}^m \) by \( g(u) = b + G(u - a) \), we have \( g(a) = b \) and

\[ c = f(u, g(u)), \quad \forall u \in \mathbb{R}^n. \]

Here’s the full nonlinear version: Suppose \( f : \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^n \) is continuously differentiable. Suppose \( p = (a, b) \in \mathbb{R}^{n+m} \) and the last \( m \) columns of \( \partial f/\partial x(p) \) form an invertible matrix. Then there is a neighbourhood \( \mathcal{U} \) of \( a \) in \( \mathbb{R}^n \) such that there is a unique continuously differentiable \( g : \mathcal{U} \longrightarrow \mathbb{R}^m \) such that \( g(a) = b \) and

\[ (u, g(u)) \in f^{-1}(f(p)), \quad \forall u \in \mathcal{U}. \]

Back to the square ... We have

\[ f(x) = (\|x_1 - p_4\|^2, \|x_1 - x_2\|^2, \|x_2 - p_3\|^2); \quad f : \mathbb{R}^4 \longrightarrow \mathbb{R}^3, \]

\( n = 1, m = 3, \) and

\[ \frac{\partial f}{\partial x} = 2 \begin{bmatrix} (x_1 - p_4)^T \\ (x_1 - x_2)^T \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ -(x_1 - x_2)^T \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -(x_1 - x_2)^T \\ (x_2 - p_3)^T \end{bmatrix}. \]
Thus \( p = (p_1, p_2) = (0, 1, 1, 1) \) and
\[
\frac{\partial f}{\partial x}(p) = 2 \begin{bmatrix}
(p_1 - p_4)^T & 0 & 0 & 0 \\
(p_1 - p_2)^T & (p_1 - p_2)^T & (p_2 - p_3)^T & \end{bmatrix} = 2 \begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]
The last 3 columns form the identity matrix. Thus there is a neighbourhood \( \mathcal{U} \) of 0 in \( \mathbb{R} \) such that there is a unique continuously differentiable \( g : \mathcal{U} \rightarrow \mathbb{R}^3 \) such that \( g(0) = (1, 1, 1) \) and
\[
(u, g(u)) \in f^{-1}(f(p)), \; \forall u \in \mathcal{U}.
\]
The flexing of the square looks like this:

There’s a 1 degree-of-freedom flexibility; in this case, the \( x \)-coordinate, \( u \), of \( x_1 \).

### 6.1.2 Rigid-Body Transformations

It’s convenient to do this in \( \mathbb{C} \) instead of \( \mathbb{R}^2 \). An isometry is a distance-preserving map \( T : \mathbb{C} \rightarrow \mathbb{C} \), that is,
\[
|T(x) - T(y)| = |x - y|, \; \forall x, y.
\]
Caution: An isometry isn’t necessarily linear. An example of an isometry is \( T(x) = e^{i\theta}x + d \):

For translation, \( \theta = 0 \), and for rotation about \( c \), \( d = c - e^{i\theta}c \). Let us say that an isometry of this form is a rigid-body transformation.

Let \( \mathcal{R} \) denote the set of rigid-body transformations on \( \mathbb{R}^2 \) (aka the special Euclidean group \( SE(2) \)). Thus \( \mathcal{R} \) can be parametrized by the variables \( (d, \theta) \in \mathbb{C} \times [0, 2\pi) \), or by three real variables. The more standard way to view \( \mathcal{R} \) uses projective geometry.
A rotation on $\mathbb{R}^2$ is achieved by a rotation matrix $R$: $R^T R = I$, $\det R = 1$. And a translation is effected by a vector $d$. Thus $T(x) = Rx + d$. We can associate with $x$ in $\mathbb{R}^2$ the vector $(x,1)$ in $\mathbb{P}^2$, the **projective plane**. Then $(x,1)$ is mapped to $(Rx+d,1)$:

$$
\begin{bmatrix}
R & d \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
1
\end{bmatrix} = 
\begin{bmatrix}
Rx + d \\
1
\end{bmatrix}.
$$

In this way $R$ is associated with the set of matrices of the form

$$
\begin{bmatrix}
R & d \\
0 & 1
\end{bmatrix}.
$$

**Lemma 25** The set $\mathcal{R}$ of rigid-body transformations in $\mathbb{R}^2$ has the structure of a 3-dimensional smooth manifold.

**6.1.3 Frameworks**

We begin with an undirected graph $G(V,E)$. We might as well take the vertices to be

$$V = \{1, 2, \ldots, n\}.$$  

Then an edge can be written $(i,j)$. Along with this graph we bring in $n$ points in the plane:

$$p_1, p_2, \ldots, p_n \in \mathbb{R}^2.$$  

And we denote the ordered $n$-tuple of these points by $p$, the **configuration vector**:

$$p = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^{2n}.$$  

Then $G(p)$ denotes a graph-like structure in the plane; it has vertices at the points $p_i$ and straight-line segments $[p_i,p_j]$ just when $(i,j) \in E$. Such a structure is called a **framework**. It is merely a realization of the graph at certain points in the plane.

We shall not fix an edge. Thus the formation is subject to rigid-body rotation and translation. Let $e$ denote the number of edges, $e = \text{card}(E)$. Order the edges in some way and define the **edge function**

$$f(p) = (\ldots, \|p_i - p_j\|^2, \ldots), \ (i,j) \in E, \ f : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^e.$$
Then \( f^{-1}(f(p)) \) is the set of \( q \in \mathbb{R}^{2n} \) such that \( \mathcal{G}(q) \) and \( \mathcal{G}(q) \) have corresponding edge lengths equal.

**Example** With respect to the previous figure,

\[
f : \mathbb{R}^8 \longrightarrow \mathbb{R}^4
\]
\[
f(p) = (\|p_1 - p_2\|^2, \|p_2 - p_3\|^2, \|p_2 - p_4\|^2, \|p_3 - p_4\|^2)
\]

We say \( p, q \in \mathbb{R}^{2n} \) are **congruent** if

\[
(\exists T \in \mathcal{R})(\forall i) Tp_i = q_i.
\]

Thus \( p, q \) are congruent iff \( \mathcal{G}(p) \) can be transformed to \( \mathcal{G}(q) \) by a rigid-body transformation.

Define \( \mathcal{M}_p \) to be the set of configurations \( q \) congruent to \( p \).

**Lemma 26** \( \mathcal{M}_p \) is a smooth (infinitely differentiable) manifold in \( \mathbb{R}^{2n} \).

The dimension of \( \mathcal{M}_p \) is of interest. Let us consider the case \( n = 3 \). Then \( \mathcal{M}_p \) is the set of vectors

\[
(Tp_1, Tp_2, Tp_3) \in \mathbb{R}^6
\]
as \( T \) ranges over all rigid-body transformations. Equivalently, the set

\[
(Rp_1 + d, Rp_2 + d, Rp_3 + d) \in \mathbb{R}^6
\]
as \( R \) ranges over all positive rotation matrices and \( d \) over all vectors. Bring in the associated projective plane. Then \( \mathcal{M}_p \) is the set of columns of the matrix

\[
\begin{bmatrix}
R & d \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
p_1 \\
p_2 \\
p_3
\end{bmatrix}
\]
as \( R \) and \( d \) vary. It turns out that if

\[
\text{rank} \begin{bmatrix}
p_1 \\
p_2 \\
p_3
\end{bmatrix} = 3,
\]

then the dimension of the set of columns of

\[
\begin{bmatrix}
R & d \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
p_1 \\
p_2 \\
p_3
\end{bmatrix}
\]
as \( R \) and \( d \) vary equals the dimension of \( \mathcal{R} \), namely 3.

**Lemma 27** Let \( n > 2 \). If the affine span of \( p_1, \ldots, p_n \) equals \( \mathbb{R}^2 \), then \( \dim \mathcal{M}_p = 3 \).

**Lemma 28** \( \mathcal{M}_p \) is a subset of \( f^{-1}(f(p)) \).
Proof Let \( q \in \mathcal{M}_p \), that is, let \( q \) be congruent to \( p \). Then \( \mathcal{G}(q) \) can be obtained from \( \mathcal{G}(p) \) by a rigid-body transformation. Obviously, \( \mathcal{G}(q) \) therefore has the same corresponding link lengths as \( \mathcal{G}(p) \). Thus \( f(q) = f(p) \).

In general the level set \( f^{-1}(f(p)) \) is not a manifold—it doesn’t have constant dimension everywhere.

A **flexing** of a framework \( \mathcal{G}(p) \) is a smooth mapping \( q : [0, 1] \rightarrow \mathbb{R}^{2n} \) with the following properties:

1. \( q(0) = p \)
2. \( (\forall t \in [0, 1]) \ q(t) \in f^{-1}(f(p)) \)
3. \( (\forall t \in (0, 1]) \ q(t) \) is not congruent to \( p \), i.e., \( q(t) \notin \mathcal{M}_p \).

A framework \( \mathcal{G}(p) \) is **flexible** if there exists a flexing of it. Otherwise it’s **rigid**. Thus \( \mathcal{G}(p) \) is flexible iff its vertices can be smoothly moved from \( p \) to noncongruent configurations while preserving all edge lengths.

**Theorem 36** \( \mathcal{G}(p) \) is rigid iff there is a neighbourhood \( \mathcal{U} \) of \( p \) such that

\[
\mathcal{M}_p \cap \mathcal{U} = f^{-1}(f(p)) \cap \mathcal{U}.
\]

Intuitively, \( \mathcal{M}_p \) and \( f^{-1}(f(p)) \) coincide near \( p \), so there’s no room for a flexing \( q(t) \).

**Examples**
Collapsed triangle—rigid or flexible?

Triangle inside a triangle—rigid or flexible?

In general it’s not easy to check rigidity. There is an easy case, however, which we look at next. The Jacobian \( \partial f/\partial x \) is an \( e \times 2n \) matrix.

Example

Consider all three vertices to be variable and take

\[
f(x) = (\|x_1 - x_2\|^2, \|x_2 - x_3\|^2).
\]

Then

\[
\frac{\partial f}{\partial x} = 2 \begin{bmatrix} (x_1 - x_2)^T & (x_2 - x_1)^T & 0 \\ 0 & (x_2 - x_3)^T & (x_3 - x_2)^T \end{bmatrix}.
\]

Thus the rank of \( \partial f/\partial x \) could be 0, 1, or 2, depending on the point \( x \). The maximum value of the rank of \( \partial f/\partial x \) over all \( x \) is called the generic rank of \( \partial f/\partial x \), denoted \( r_g \); in this example \( r_g = 2 = e \). A point \( p \) is regular if the rank of \( \partial f/\partial x(p) \) equals the generic rank. In this example, these are the regular points:

\[
p = (p_1, p_2, p_3), \quad p_1 \neq p_2 \& p_2 \neq p_3.
\]

Notice that almost all points are regular.

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The next result can be proved using the Implicit Function Theorem.

Lemma 29 Suppose \( p \) is a regular point. Then there is an open neighbourhood \( U \) of \( p \) such that \( f^{-1}(f(p)) \cap U \) is a smooth manifold of dimension \( 2n - r_g \).

Theorem 37 Suppose \( p \) is a regular point and the affine span of \( p_1, \ldots, p_n \) is \( \mathbb{R}^2 \). Then \( G(p) \) is rigid iff

\[
\text{rank } \frac{\partial f}{\partial x}(p) = 2n - 3.
\]

Proof \( G(p) \) is rigid

\[
\iff (\exists U) M_p \cap U = f^{-1}(f(p)) \cap U
\]

\[
\iff (\exists U) \dim(M_p \cap U) = \dim(f^{-1}(f(p)) \cap U)
\]

\[
\iff 3 = 2n - r_g
\]

\[
\iff r_g = 2n - 3
\]

\[
\iff \text{rank } \frac{\partial f}{\partial x}(p) = 2n - 3.
\]

6.1.4 Infinitesimal Rigidity

Now we turn to a simpler, “linearized” concept of rigidity. Given a framework \( G(p) \). Assume \( G(p) \) is flexible. Then there’s a flexing \( q(t) \in f^{-1}(f(p)) \). We can differentiate and look at the tangent to the curve:

![Diagram of infinitesimal rigidity](image)

\[
\frac{\dot{q}(0)}{q(0)}
\]

Example
Link constraints:
\[ \|p_1 - p_2\| = 1, \quad \|p_2 - p_3\| = 1. \]

Given configuration:
\[ p_1 = (0,0), \quad p_2 = (1,0), \quad p_3 = (1,1). \]

The flexing satisfies
\[ [q_1(t) - q_2(t)]^T [q_1(t) - q_2(t)] = 1 \]
\[ [q_2(t) - q_3(t)]^T [q_2(t) - q_3(t)] = 1. \]

Take \( d/dt \) and denote \( \dot{q} \) by \( v \):
\[ 2[q_1(t) - q_2(t)]^T [v_1(t) - v_2(t)] = 0 \]
\[ 2[q_2(t) - q_3(t)]^T [v_2(t) - v_3(t)] = 0. \]

Evaluate at \( t = 0 \):
\[ 2(p_1 - p_2)^T [v_1(0) - v_2(0)] = 0 \]
\[ 2(p_2 - q_3)^T [v_2(0) - v_3(0)] = 0. \]

That is, \( R(p) v(0) = 0 \), where
\[
R(p) = 2 \begin{bmatrix}
(p_1 - p_2)^T & -(p_1 - p_2)^T & 0 \\
0 & (p_2 - p_3)^T & -(p_2 - p_3)^T
\end{bmatrix}.
\]

Notice that \( R(p) = \partial f/\partial x(p) \) for the edge function
\[ f(x) = (\|x_1 - x_2\|^2, \|x_2 - x_3\|^2); \quad f : \mathbb{R}^6 \rightarrow \mathbb{R}^2. \]

The matrix \( R(p) \) is called the **rigidity matrix**. Here
\[
R(p) = \begin{bmatrix}
-1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0
\end{bmatrix}.
\]

Now rank \( R(p) = 2 \), so \( \dim \ker R(p) = 4 \).

Thus there are 4 linearly independent infinitesimal motions: 3 of these correspond to rigid motion (\( x \)-translation, \( y \)-translation, rotation), and the 4th corresponds to infinitesimal flexibility—there’s 1 infinitesimal degree-of-freedom. The example is thus infinitesimally flexible. (It is also flexible.)

We saw in our study of rigidity that \( \mathcal{M}_p \) is contained in \( f^{-1}(f(p)) \). The linearized version of this is as follows.

**Lemma 30** The tangent space \( T_p \mathcal{M}_p \) is a subspace of \( \ker R(p) \):
\[ T_p \mathcal{M}_p \subset \ker R(p). \]
An infinitesimal flexing of a framework $\mathcal{G}(p)$ is a nonzero vector $v \in \mathbb{R}^{2n}$ such that $v \in \ker R(p)$ but $v \notin T_p\mathcal{M}_p$.

A framework $\mathcal{G}(p)$ is infinitesimally flexible if there exists an infinitesimal flexing of it. Otherwise it’s infinitesimally rigid. Thus

$$\text{inf rigid} \iff T_p\mathcal{M}_p = \ker R(p) \iff \dim(T_p\mathcal{M}_p) = \dim(\ker R(p)) \iff \dim(T_p\mathcal{M}_p) = 2n - \text{rank } R(p) \iff \text{rank } R(p) = 2n - \dim(T_p\mathcal{M}_p) \iff \text{rank } R(p) = 2n - \dim(\mathcal{M}_p).$$

We saw that if the affine span of $p_1, \ldots, p_n$ equals $\mathbb{R}^2$, then $\dim \mathcal{M}_p = 3$. Thus

**Theorem 38** Assume the affine span of $p_1, \ldots, p_n$ equals $\mathbb{R}^2$. Then $\mathcal{G}(p)$ is infinitesimally rigid iff $\text{rank } R(p) = 2n - 3$.

Compare Theorems 37 and 38.

Finally:

**Theorem 39** Assume the affine span of $p_1, \ldots, p_n$ equals $\mathbb{R}^2$. Then $\mathcal{G}(p)$ is infinitesimally rigid iff $\mathcal{G}(p)$ is rigid and $p$ is regular.

**Proof** ($\iff$) From Theorem 37, $\text{rank } R(p) = 2n - 3$. Then from Theorem 38, $\mathcal{G}(p)$ is infinitesimally rigid.
(⇒) $G(p)$ is rigid, since flexibility implies infinitesimal flexibility. So it remains to show $p$ is regular. Near $p$ is a regular point $q$ such that the affine span of $q_1, \ldots, q_n$ equals $\mathbb{R}^2$. By Lemma 30
\[ T_q M_q \subset \ker R(q), \]
and so
\[ 3 \leq 2n - \text{rank } R(q), \]
that is, $\text{rank } R(q) \leq 2n - 3$. Since $q$ is regular, $r_g \leq 2n - 3$. But by Theorem 38, $\text{rank } R(p) = 2n - 3$. Therefore $\text{rank } R(p) = r_g$, so $p$ is regular. \qed

**Example** An example that's rigid but infinitesimally flexible:

\[ \begin{array}{c}
\includegraphics[width=0.5\textwidth]{example.png}
\end{array} \]

**6.1.5 Exercises**

1. Consider a collapsed triangle, with vertices
\[ p_1 = (1,0), \quad p_2 = (2,0), \quad p_3 = (0,0). \]
Take the edge function
\[ f(x_1) = (\|x_1 - p_2\|^2, \|x_1 - p_3\|^2). \]
Calculate the rank of $\partial f / \partial x_1(p_1)$. What can you conclude?

2. Find the general form of a reflection about a fixed line. Is this an isometry? A rigid-body transformation?

3. Find the rigid-body transformation that does this:

\[ \begin{array}{c}
\includegraphics[width=0.3\textwidth]{exercises.png}
\end{array} \]

centre at (1,1) centre at (1,0.5)
4. The **affine span** of the vectors $p_1, \ldots, p_n$ is defined to be the set

$$\{\lambda_1 p_1 + \cdots + \lambda_n p_n : \lambda_i \in \mathbb{R}, \lambda_1 + \cdots + \lambda_n = 1\}.$$ 

(a) Sketch the affine span of two vectors, $p_1, p_2$.

(b) Prove that the affine span of $p_1, \ldots, p_n$ equals $\mathbb{R}^2$ iff

$$\text{rank} \begin{bmatrix} p_1 & \cdots & p_n \\ 1 & \cdots & 1 \end{bmatrix} = 3.$$ 

(c) Show that if $n > 2$, then the affine span of $p_1, \ldots, p_n$ equals $\mathbb{R}^2$ if the vectors are selected using a random number generator. That is, the condition is generic.

5. Consider the collapsed triangle with three links. Is $p$ regular? Is the affine span of $p_1, p_2, p_3$ equal to $\mathbb{R}^2$? Is the framework rigid? Is the framework infinitesimally rigid?

6. Answer the question we began with: For a formation of $n$ point robots, how many distances need to be maintained for the whole formation to be in equilibrium?

6.1.6 References


2. I. Streinu, “A crash course in rigidity theory,” online presentation.


6.2 Polygon Formations

This section is the MASc work of Laura Krick.

In the first half of the course the focus was on the rendezvous problem for point robots. In this section the task is to achieve a geometric formation. An early reference is


6.2.1 Problem Statement

Consider $n$ robots moving in the plane, $\mathbb{R}^2$. They are modeled as usual by $\dot{z}_i = u_i$, where $z_i(t)$ is the position of robot $i$ and $u_i(t)$ is its velocity input. The goal is to have the points $z_1(t), \ldots, z_n(t)$ converge to form a stable equilibrium formation. This requirement could be described in general terms, but we prefer a specific example in order to derive explicit formulas. Thus, the desired formation is taken to be an ordered regular polygon with vertices in the order $z_1, z_2, \ldots, z_n$ and with sides a prescribed length $d$. Here’s an example formation for six robots:

![Polygon Formation Diagram](image)

We call such an arrangement a $d$-polygon.

Additionally, we assume the robots have only onboard sensors. They can sense relative positions of other robots but not absolute positions. With this setup, we have the following problem statement: Given a distance $d$, design control laws so that $d$-polygons are stable equilibrium formations. It’s possible to argue that the controllers must be nonlinear.

As in the reference cited above, our solution begins with rigid graph theory, which is the subject of the preceding section. But first, a pause to review Jacobians.

Review of Jacobians

Let $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$. The Jacobian of $f$ at the point $x$ is the $n \times m$ matrix $J_f(x)$ that satisfies

$$\frac{d}{d\varepsilon} f(x + \varepsilon h) \bigg|_{\varepsilon=0} = J_f(x)h.$$ 

The left side is the derivative of $f$ at $x$ in the direction $h$. This turns out to be a linear transformation of $h$, namely, $h \mapsto J_f(x)h$.  

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Thus for example \( f(x) = x(\|x\|^2 - d^2) \), \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \):

\[
\begin{align*}
f(x + \varepsilon h) &= (x + \varepsilon h)(\|x + \varepsilon h\|^2 - d^2) \\
&= (x + \varepsilon h)(\|x\|^2 + 2\varepsilon x^T h + \varepsilon^2 \|h\|^2 - d^2) \\
&= x\|x\|^2 + \varepsilon \|x\|^2 h + 2\varepsilon xx^T h - \varepsilon d^2 h + \text{higher order terms}.
\end{align*}
\]

So
\[
J_f(x) = (\|x\|^2 - d^2) I + 2xx^T.
\]

6.2.2 Example: 3 Robots

Let’s study three robots trying to form an equilateral triangle. The robots are located at \( z_1, z_2, z_3 \) and the relative displacements are

\[
e_1 = z_2 - z_1, \quad e_2 = z_3 - z_2, \quad e_3 = z_1 - z_3.
\]

Thus \( e = Pz \) for the incidence matrix

\[
P = \begin{bmatrix}
-1 & 1 & 0 \\
0 & -1 & 1 \\
1 & 0 & -1
\end{bmatrix} \otimes I.
\]

The aggregate vectors \( z \) and \( e \) live in the aggregate space \( (\mathbb{R}^2)^3 \). We define the vector of norms

\[
v(e) = (\|e_1\|^2, \|e_2\|^2, \|e_3\|^2).
\]

Thus \( v : (\mathbb{R}^2)^3 \rightarrow \mathbb{R}^3 \) and the function \( v(Pz) \) is the edge function of the preceding section. If \( z \) is regular (i.e., the triangle has positive area), the triangle is infinitesimally rigid and the rank of the Jacobian of the edge function equals \( 2n - 3 = 3 \).

In the \( d \)-triangle formation, \( v(e) = d^2 \mathbf{1} \) (where \( \mathbf{1} \) is the vector of three 1’s) so we define the potential function

\[
\phi(z) = \frac{1}{2} \|v(Pz) - d^2 \mathbf{1}\|^2.
\] (6.1)

This potential function is just one candidate. It is the one induced by the rigid graph, and it does have the important property that \( \phi(z) = 0 \) if and only if the positions \( z_i \) form a \( d \)-triangle. The proposed control law is the negative gradient of the potential function: \( u = -J_\phi(z)^T \).

An explicit formula for \( u \) can be derived as follows. We have from (6.1)

\[
J_\phi(z) = [v(Pz) - d^2 \mathbf{1}]^T J_v(Pz) P.
\]

Now

\[
J_v(e)^T = 2 \begin{bmatrix}
e_1 & 0 & 0 \\
0 & e_2 & 0 \\
0 & 0 & e_3
\end{bmatrix}.
\] (6.2)

Defining the vector

\[g(e) = J_v(e)^T [v(e) - d^2 \mathbf{1}]\]

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with components

\[ g_i(e) = 2(\|e_i\|^2 - d^2)e_i, \]

we can write \( u \) in the compact form

\[ u = -P^Tg(e). \]

More explicitly, the equations for the controlled three robots are

\[
\begin{align*}
\dot{z}_1 &= 2(\|z_2 - z_1\|^2 - d^2)(z_2 - z_1) + 2(\|z_3 - z_1\|^2 - d^2)(z_3 - z_1) \\
\dot{z}_2 &= 2(\|z_1 - z_2\|^2 - d^2)(z_1 - z_2) + 2(\|z_3 - z_2\|^2 - d^2)(z_3 - z_2) \\
\dot{z}_3 &= 2(\|z_1 - z_3\|^2 - d^2)(z_1 - z_3) + 2(\|z_2 - z_3\|^2 - d^2)(z_2 - z_3).
\end{align*}
\]

Thus each robot looks at the other two.

Now we turn to formations and stability. The equation in \( z \)-space is

\[
\dot{z} = -P^Tg(Pz) = -P^TJ_v(Pz)^T[v(Pz) - d^21] \tag{6.3}
\]

and in \( e \)-space it is

\[
\dot{e} = P\dot{z} = -PP^Tg(e) = -PP^TJ_v(e)^T[v(e) - d^21]. \tag{6.4}
\]

**Lemma 31** The centroid of the robot positions is stationary.

**Proof** Bring in \( 1 \otimes I \), the \( 6 \times 2 \) matrix

\[
\begin{bmatrix}
I \\
I \\
I
\end{bmatrix}.
\]

The centroid is \( \frac{1}{3}(1 \otimes I)^Tz \). Multiplying (6.3) by \( 1 \otimes I \) and using \( (1 \otimes I)^TP^T = 0 \) give that \( (1 \otimes I)^Tz(t) \) is constant. \( \square \)

The matrix \( PP^T \) equals

\[
\begin{bmatrix}
2 & -1 & -1 \\
-1 & 2 & -1 \\
-1 & -1 & 2
\end{bmatrix} \otimes I.
\]

The left-hand matrix is the Laplacian of the triangle viewed as an undirected graph.
**Linearized Model**

We are interested in the equilibria of (6.3) with $v(Pz) = d^2 \mathbf{1}$, that is, the $d$-triangle, though there are other equilibria. Let $\mathbf{z}$ be a triangle equilibrium. Write (6.3) in the form $\dot{z} = f(z)$, where

$$f(z) = -P^T J_{v}(Pz)^T [v(Pz) - d^2 \mathbf{1}].$$

Then the linearized system is

$$\dot{\delta z} = J_f(\mathbf{z}) \delta z.$$

Define

$$G = 4 \begin{bmatrix} \overline{e_1}e_1^T & 0 & 0 \\ 0 & \overline{e_2}e_2^T & 0 \\ 0 & 0 & \overline{e_3}e_3^T \end{bmatrix}.$$  \hspace{1cm} (6.5)

Then a calculation gives

$$J_f(\mathbf{z}) = -P^T GP$$

and so

$$\dot{\delta z} = -P^T GP \delta z.$$

It is interesting to show the connection between the matrix $-P^T GP$ of the linearized system and the edge function of the triangle, namely, $f_{edg}(z) := v(Pz)$. We have

$$J_{f_{edg}}(z) = J_{v}(Pz)P.$$

Thus from (6.2) and (6.5),

$$J_{f_{edg}}(z)^T J_{f_{edg}}(z) = P^T GP.$$  

This equation gives a connection between graph rigidity and control dynamics. For example:

**Lemma 32** The matrix $-P^T GP$ has rank 3, hence 3 zero eigenvalues; the other 3 eigenvalues are real and negative.

Here’s the main result that the triangle is locally asymptotically stable.

**Theorem 40** Let $\mathbf{z}$ be a triangle equilibrium point of (6.3). There exists an open ball $B$ centred at $\mathbf{z}$ such that if $z(0) \in B$, then $z(t)$ converges asymptotically to a nearby triangle formation.

The picture is this:
The proof is a tour de force of analysis—ask Laura if you’re interested.

6.2.3 Example: 6 Robots

For three robots, each robot ends up seeing all the others. The case of 6 robots is different: The visibility graph will not need to be complete.

Here are the steps:

1. Form an infinitesimally rigid graph for 6 robots in a regular \( d \)-polygon. Some link lengths will be different from \( d \).

2. Form the potential function \( \phi(z) \) from these links are their desired lengths.

3. Use the control law \( u = -J\phi(z)^T \).

The resulting regular \( d \)-polygon formations will be locally asymptotically stable.

Laura took this graph:
Each robot sees four others. The graph has 12 links. The complete graph would have 15 links, and 10 links are the minimum needed for global rigidity. Here’s a simulation showing convergence to a regular polygon starting from random initial positions:

![Simulation of polygon formation](chart)

The polygon formation is not globally stable, because there are other equilibria.
6.3 Water Tank Networks

Water tank networks are interesting to study from the point of view of rendezvous. Consider a tank of water:

Let the water level be denoted $x$. Then the pressure at the outlet is proportional to $\sqrt{x}$, so the flow rate out is proportional to $\sqrt{x}$ (Torricelli’s Law); for simplicity, let the proportionality constant be 1. Then the continuous-time model is

$$\dot{x} = u, \quad u = -\sqrt{x}, \quad x \geq 0.$$

Now consider two identical coupled tanks:

The model is

$$\dot{x}_1 = u_1$$
$$\dot{x}_2 = u_2,$$

where

$$u_1 = \begin{cases} -\sqrt{x_1 - x_2}, & x_1 \geq x_2 \\ \sqrt{x_2 - x_1}, & x_1 < x_2 \end{cases}$$
$$u_2 = \begin{cases} -\sqrt{x_2 - x_1}, & x_2 \geq x_1 \\ \sqrt{x_1 - x_2}, & x_2 < x_1. \end{cases}$$

Refer $x_2$ to $x_1$ by defining the difference $d = x_2 - x_1$. Then the model for $d$ is

$$\dot{d} = f(d),$$

where

$$f(d) = \begin{cases} 2\sqrt{-d}, & d \leq 0 \\ -2\sqrt{d}, & d > 0. \end{cases}$$
So $f$ is continuous but not Lipschitz.\footnote{Many references say that Torricelli’s law is only an approximation anyway, and they go ahead and linearize somehow.}

Of course, we can solve (6.6): If $d(0) > 0$, then

$$d(t) = \begin{cases} 
\left[\sqrt{d(0)} - t\right]^2, & t \leq \sqrt{d(0)} \\
0, & t > \sqrt{d(0)}. \end{cases}$$

In this way, (6.6) has a unique solution and $d(t)$ converges to 0 in finite time for every initial condition. Thus the two water levels rendezvous.

It would be nice to extend this to more tanks:

And then to time-varying coupling: