Theorems on Algebraic Operations on Series: Let \( \sum a_n \) and \( \sum b_n \) be any two series.

1. If \( \sum a_n \) and \( \sum b_n \) both converge, then \( \sum (a_n \pm b_n) \) must converge.
2. If \( \sum a_n \) converges, and \( C \) is a real number, then \( \sum Ca_n \) must converge.
   If \( \sum a_n \) diverges, and \( C \) is a real number, then \( \sum Ca_n \) must diverge.
3. If one of \( \sum a_n \) or \( \sum b_n \) converges and the other diverges, then \( \sum (a_n \pm b_n) \) must diverge.

TESTS FOR CONVERGENCE/DIVERGENCE

Geometric Series:
Let \( a \) and \( r \) be real numbers. A series of the form \( a + ar + ar^2 + \ldots + ar^n + \ldots = \sum_{n=0}^{\infty} ar^n \) is called a geometric series. Geometric series converges to \( \frac{a}{1-r} \) if \( |r| < 1 \), and diverges if \( |r| \geq 1 \).

Harmonic Series and \( p \)-Series:
The series \( \sum_{n=1}^{\infty} \frac{1}{n} \) is called the harmonic series and it diverges. The \( p \)-series has the form \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) and it converges if \( p > 1 \) and diverges if \( p \leq 1 \).

The \( n \)-th Term Test (The Test for Divergence):
Let \( \sum a_n \) be any series. If \( \lim_{n \to \infty} a_n \neq 0 \) then \( \sum a_n \) must diverge.
(note: if \( \lim_{n \to \infty} a_n = 0 \), no conclusion can be made)

The Integral Test:
Let \( f \) be continuous and decreasing on \([k, \infty)\) such that \( f(x) \geq 0 \) on \([k, \infty)\). Let \( a_n = f(n) \), then:
\[
\sum_{n=k}^{\infty} a_n \text{ converges iff } \int_{k}^{\infty} f(x) \, dx \text{ converges.}
\]
\[
\sum_{n=k}^{\infty} a_n \text{ diverges iff } \int_{k}^{\infty} f(x) \, dx \text{ diverges.}
\]

The Direct Comparison Test:
Let \( \sum a_n \) and \( \sum b_n \) be any two series such that \( a_n \geq 0 \) and \( b_n \geq 0 \) for all (large) \( n \).
If \( a_n \leq b_n \) for all (large) \( n \) and if \( \sum b_n \) converges, then \( \sum a_n \) must converge.
If \( a_n \geq b_n \) for all (large) \( n \) and if \( \sum b_n \) diverges, then \( \sum a_n \) must diverge.
Remember it as: if the smaller series diverges, the larger series must diverge, and if the larger series converges, the smaller series must converge.
(note: no conclusion can be made if the “smaller” series converges or the “larger” series diverges: in this case, try using the Limit Comparison Test)
The Limit Comparison Test:
Let $\sum a_n$ and $\sum b_n$ be any two positive series. If $\lim_{n \to \infty} \frac{a_n}{b_n} = C$, where $C$ is a finite number $\neq 0$, then:

$\sum a_n$ converges iff $\sum b_n$ converges.

$\sum a_n$ diverges iff $\sum b_n$ diverges.

To choose an appropriate $\sum b_n$, look at the behaviour of $\sum a_n$ for large $n$, take the highest power of $n$ in the numerator and denominator (ignoring coefficients) and simplify:

For example, if $\sum a_n = \frac{5n^3 - n + 2}{2n^5 - 3n^2 + n - 1}$, then at large $n$, the $n^3$ and $n^5$ terms “dominates”, resulting in $\sum b_n = \frac{n^3}{n^5} = \frac{1}{n^2}$ (note the omission of the coefficients).

(note: it happens that putting $b_n$ in the denominator usually makes the algebra easier – but ultimately it doesn’t matter if you’re taking the $\lim_{n \to \infty} \frac{a_n}{b_n}$ or $\lim_{n \to \infty} \frac{b_n}{a_n}$)

The Alternating Series Test:
Def/ An alternating series is a series whose terms alternate in sign.
Ex. $-1 + 2 - 3 + 4 - \ldots$

Let $\sum a_n$ be any alternating series. If $|a_n| \geq |a_{n+1}|$ for all $n$, and if $\lim_{n \to \infty} |a_n| = 0$, then the series must converge.

Remember this by: an alternate series only converges if its $n$th term converges to zero, and its’ terms are non-increasing (ie. ignoring minus signs, each term is smaller than or same as its predecessor).

(note: this test only tells if the alternating series converges – it tells you NOTHING about the positive-term series; also if the series fails the “non-increasing” condition of this test, no conclusion can be made about convergence or divergence of the series)

Tip: ALWAYS check the $n$th term first, because the series and its positive term series diverge if the $n$th term is not zero. Also, to tell whether a function decreases or increases, you can use the First Derivative Test.

Absolute and Conditional Convergence
Let $\sum a_n$ be any series. $\sum a_n$ converges absolutely if $\sum |a_n|$ converges. If $\sum a_n$ converges absolutely, then the series $\sum a_n$ itself must converge.

Let $\sum a_n$ be any series. $\sum a_n$ converges conditionally if $\sum a_n$ converges but $\sum |a_n|$ diverges.

The Root Test
Let $\sum a_n$ be any series. Suppose $\lim_{n \to \infty} \sqrt[n]{|a_n|} = L$, then:

If $0 \leq L < 1$, $\sum a_n$ converges absolutely.

If $L > 1$ (including $L = +\infty$), $\sum a_n$ diverges.

If $L = 1$, no conclusion can be made.

The Ratio Test
Let $\sum a_n$ be any series of non-zero terms. Suppose $\lim_{n \to \infty} |\frac{a_{n+1}}{a_n}| = L$, then:

If $0 \leq L < 1$, $\sum a_n$ converges absolutely.

If $L > 1$ (including $L = +\infty$), $\sum a_n$ diverges.

If $L = 1$, no conclusion can be made.

*This test is very useful if $a_n$ contains a factorial or if you are dealing with power series.

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