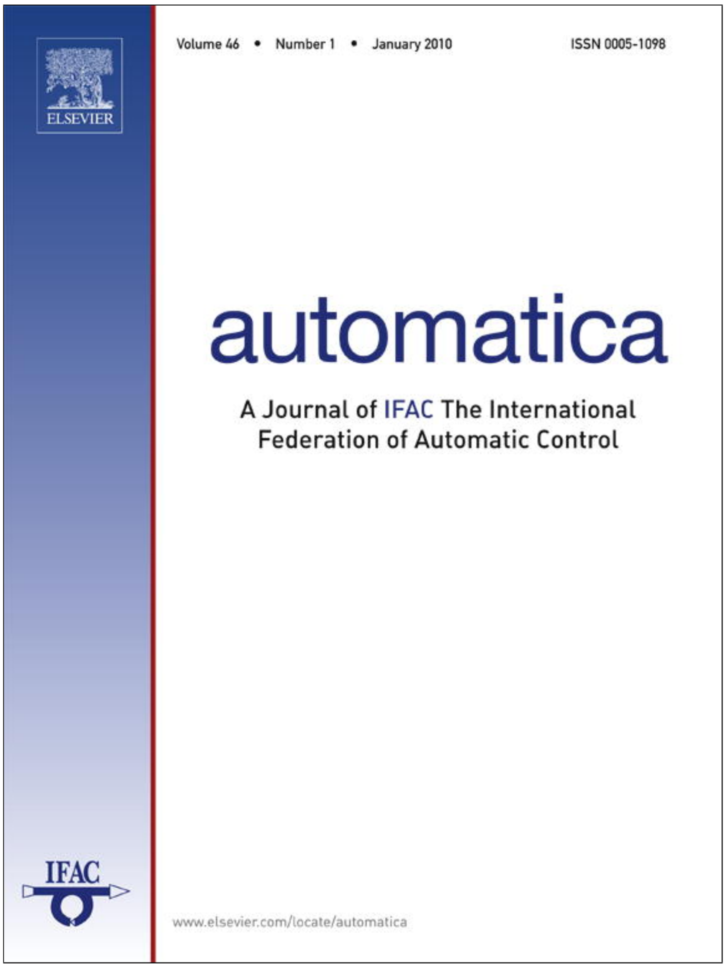
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Brief paper

Collective motions and formations under pursuit strategies on directed acyclic graphs ☆

Wei Ding, Gangfeng Yan, Zhiyun Lin *

Asus Intelligent Systems Lab, Department of Systems Science and Engineering, Zhejiang University, 38 Zheda Road, Hangzhou, 310027, PR China

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ABSTRACT

A novel pursuit-based approach is presented to investigate collective motions and formations of a large number of agents with both single-integrator kinematics and double-integrator dynamics on directed acyclic graphs (DAGs). Each agent pursues its neighbors according to a directed acyclic graph, in which the agents without neighbors are leaders. Based on signal flow graph analysis and Mason's rule, necessary and sufficient conditions are derived for BIBO stability of resulting pursuit systems. Moreover, achievable collective motions and formations are analyzed by adjusting a set of control parameters when leaders keep stationary, perform uniform rectilinear motions, and perform uniform circular motions. Finally, simulations are provided for achieving a static formation and mimicking several complex collective behaviors observed in nature, such as V-formation, vortex motions, and tornado motions.

1. Introduction

In nature, many animal species aggregate to live in groups for the benefit of avoiding predators and in order to efficiently find food (Krause & Ruxton, 2002). It is observed that swarm members (or agents) interact each other locally, yet desired collective behaviors are achieved (Sumpter, 2006). Thus, fundamental questions about what local interaction rules are and how they work, are raised, and attract significant interest amongst biologists (e.g., Couzin & Krause, 2003; Parrish, Viscido, & Grunbaum, 2002), physicists (e.g., Gönci, Nagy, & Vicsek, 2008), and control engineers (e.g., Jadbabaie, Lin, & Morse, 2003; Lin, 2008). Local interaction rules have various interpretations and implementations, such as Vicsek's model (Jadbabaie et al., 2003; Vicsek, Czirók, Ben-Jacob, Cohen, & Shochet, 1995), oscillator models (Paley, Leonard, Sepulchre, Grünbaum, & Parrish, 2007; Sepulchre, Paley, & Leonard, 2008), and artificial potential based approaches (Chen & Leung, 2006; Chu, Wang, Chen, & Mu, 2006; Olfati-Saber, 2006; Shi, Wang, & Chu, 2006; Tanner, Jadbabaie, & Pappas, 2007), etc. Inspired by the progress in the field, this paper tries to present a new approach based on pursuit strategies to investigate collective motions and formations of a large number of agents.

The history related to pursuit problems can be dated back as early as 1732 when the mathematics of pursuit curves was first studied by French scientist Pierre Bouguer. Recently, cyclic pursuit algorithms have been investigated a lot (e.g. Lin, Broucke, & Francis, 2004; Marshall, Broucke, & Francis, 2004; Sinha & Ghose, 2007) for the purpose of formation control, since Bruckstein, Cohen, and Efrat (1991) explored the collective behaviors of ants, crickets, and frogs. More recently, Pavone and Frazzoli (2007) let each agent pursue its predecessor along the line of sight rotated by an offset angle in a cyclic pursuit and assumed that the offset angles for all agents are identical. Consequently, the poles of the system in Pavone and Frazzoli (2007) are rotated by the offset angle in the complex plane compared to the ones in (Lin et al., 2004). Thus, rendezvous, uniform circular motions, and logarithmic spiral motions are achieved depending on the offset angle. By extending from the unidirectional ring topology in Pavone and Frazzoli (2007), Ren (2008a,b) proved a result similar to that in Pavone and Frazzoli (2007): given a network topology containing a spanning tree and also satisfying certain other technical conditions, if the offset angle is below, equal, or above a critical value, all agents eventually rendezvous, move on circular orbits, or diverge. However, unlike rendezvous behaviors, the more interesting case of circular motions is marginally stable and inherently unrobust due to the critical value for the offset angle. On the other hand, Ding, Yan, and Lin (2009) extended the result of Pavone and Frazzoli (2007) to a hierarchical cyclic pursuit to attain different kinds of circular motions, but again, these motions are not robust for the same reason. This paper also takes inspiration from Pavone and Frazzoli (2007), but studies pursuit strategies under directed

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* Corresponding author. Tel.: +: +86 571 8795 1637; fax: +86 571 8795 2152.
E-mail address: linz@zju.edu.cn (Z. Lin).

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acyclic graphs where the offset angles and the gains are not restricted to being equal. Directed acyclic graphs are a general class of graphs, which have been used in the leader-following architecture (e.g. Das et al., 2002; Tanner, Pappas, & Kumar, 2004). For example, birds do not look backward to form a cycle in their interaction topology when flying in a V-formation. Also, in military applications like logistical transportation in the battlefield using a platoon of vehicles, it may be expected that only few are manned vehicles act as leaders and the others are unmanned ones. While moving as a group from one place to another, each unmanned vehicle follows its neighbors using the information from its onboard sensors that are only capable of looking forward (e.g., cameras) and thus the platoon forms a directed acyclic graph. Moreover, Couzin, Krause, Franks, and Levin (2005) indicated that in nature, only a minority of agents take the roles of leaders and determine the motion of the group. This motivates us to study the directed acyclic topology and to explore what kinds of motions can be achieved with pursuit strategies under directed acyclic graphs. A directed acyclic graph may have two or more leaders, which means it may not have a spanning tree as required in Ren (2008a,b). Also, each agent may have a different offset angle and gain from the others in our setup. Utilizing different offset angles and gains is of a great advantage because of one aspect, a team of distributed agents do not need to agree on a common parameter to achieve a collective motion and because of the other aspect, it provides many more degrees of freedom in control design so that any desired formation can be attained and maintained stably to meet the requirements of engineering applications. Moreover, since the offset angles are not identical for all agents in the paper, the standard analysis techniques used in Pavone and Frazzoli (2007), Ren (2008a,b) and Ding et al. (2009) by checking how the poles are rotated by the offset angle in the complex plane are not applicable. In the paper, we propose significantly different analysis tools based on signal flow graphs and Mason's rule.

We consider both single-integrator kinematics and double-integrator dynamics. Necessary and sufficient conditions for BIBO stability of the system under pursuit in directed acyclic graphs are derived, where the leaders' trajectories are viewed as the inputs and the follower's trajectories are treated as the outputs. The necessary and sufficient conditions we obtained only require us to check the complex degree of every follower node in the graph. In other words, every agent knows how to contribute to a stable collective motion for the group just from its own control parameters even though there is no centralized supervisor. As a result, if the goal is just a collective motion like swarming, it can be achieved in a simple distributed way, that is, each agent just selects its own offset angle and gain to satisfy the requirements placed upon it, and does not need to take into account the effects from others. However, in order to achieve a desired formation in motion, the parameters have to be carefully designed by a central entity, which is usually the case for formation control. In the paper, achievable formations are analyzed for the group when leaders keep stationary and perform uniform rectilinear motions or uniform circular motions. Thus, designing the offset angles and gains to attain a desired stable formation is straightforward. Simulations are provided for achieving a static formation and mimicking several complex collective behaviors observed in nature such as V-formation, vortex motions, and tornado motions. Our approach for formation control compared to others, is simple and computationally efficient, requires less information (only relative positions for single-integrator, and relative positions and agents' own velocity for double-integrator), and scales well to different swarm sizes, while in Lin et al. (2004) and Ren (2007a) the consensus-based formation control requires all agents to have a common sense of direction and in Shi et al. (2006), Bai, Arcak, and Wen (2008), Wang (1989), Shi, Wang, Chu, and Xu (2007) and Ren (2007b) the agents need to know the velocity and acceleration of neighbors and/or leaders.

2. Preliminary and setup

Notations: We use $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{R}^+$ to denote the set of complex numbers, real numbers, and positive real numbers, respectively. $\iota = \sqrt{-1}$ denotes the imaginary unit. $1(t)$ denotes the unit step function.

2.1. Directed acyclic graphs

A *directed graph (digraph)* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a non-empty node set $\mathcal{V} = \{1, 2, \ldots, N\}$ and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. An edge of $\mathcal{G}$ is denoted by an ordered pair of nodes, (i, j), meaning that the edge leaves node i and enters node j. Meanwhile, node i and node j are named the tail and head of edge (i, j), respectively. A *walk* is an ordered sequence of nodes such that any two consecutive nodes in the sequence correspond to an edge of the digraph. If the nodes in a walk are distinct, the walk is called a *path*. If a walk starts and ends at the same node and all other nodes on the walk are distinct, it is called a *cycle*. A digraph without cycles is a *directed acyclic graph* (DAG).

In the paper, the neighboring relationship of networked agents is schematically represented by a DAG. The agents, which are not heads of any edges in the DAG, are leaders of the group, and all other agents are followers. Obviously, an isolated agent is a trivial leader. Suppose that the indices of agents are arranged in such a way that agent i for $i \in \{1, 2, \ldots, N_l\}$ is a leader, and agent i for $i \in \{N_l + 1, N_l + 2, \ldots, N\}$ is a follower. Follower i's neighbor set is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} | (j, i) \in \mathcal{E}\}$.

2.2. Signal flow graphs

A signal flow graph (SFG) is a weighted digraph, where the nodes represent system states, the edges indicate the directions of signal flows, and the gain of each edge is a transfer function from one state to another (Mason, 1956). A node performs two functions: (1) Addition of the signals on all incoming edges; (2) Transmission of the total node signal to all outgoing edges. In an SFG, *source nodes (sources)* represent independent system states, having only outgoing edges, *sink nodes (sinks)* represent dependent system states, having only incoming edges, and *mixed nodes* have both incoming and outgoing edges. *A forward path* and *loop* in SFGs are the notions of path and cycle in digraphs, respectively. The *forward path (loop) gain* is the product of all the edge gains on the forward path (loop). Loops are *nontouching* if they have no common node. For an SFG containing a single source and a sink, the graph determinant is

$$\Delta = 1 - \sum L_1 + \sum L_2 - \sum L_3 + \cdots, \quad (1)$$

where $\sum L_1$ is the sum of all loop gains in the SFG, $\sum L_2$ is the sum of the products of gains in all possible combinations of nontouching loops taken two at a time, $\sum L_3$ is the sum of the products of gains in all possible combinations of nontouching loops taken three at a time, and so on. In addition, we denote Δ_i the determinant of the remaining subgraph when the ith forward path is removed, and call it *cofactor* of the forward path. For an SFG without loops, its determinant and all cofactors are always 1.

We finally recall *Mason's rule* to calculate the overall transfer function from a source to a sink.

Theorem 2.1 (*Mason's Rule (Mason, 1956)*). *In an SFG containing only one source and one sink, the overall gain T is given by*

$$T = \frac{\sum T_i \Delta_i}{\Delta} \quad (2)$$

where Δ is the graph determinant, T_i and Δ_i are the forward path gain and the cofactor of the ith forward path from the source to the sink, respectively.

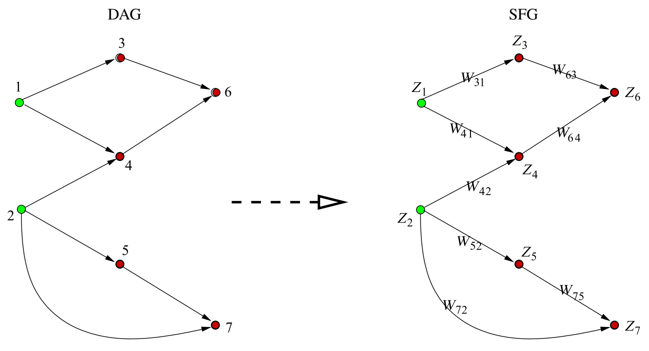

Fig. 1. From DAG to SFG.

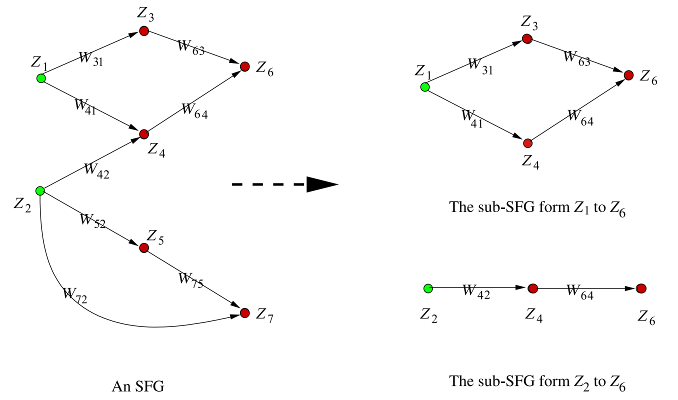

Fig. 2. Sub-SFGs from each source to node Z_6.

3. Pursuit systems (Single-integrator kinematics)

Let $z_i \in \mathbb{C}$ ($i \in \{1, 2, \ldots, N\}$) denote the position of agent i in a 2D-space. Each agent is modeled by a single-integrator kinematics

$$\dot{z}_i = u_i$$

where $u_i \in \mathbb{C}$ is the control input. The purpose of utilizing complex variables instead of conventional 2D state space representation is to simplify the analysis using signal flow graphs and Mason's rule. For a DAG describing the neighboring relationship of a network of N agents, we consider the following pursuit strategy for follower i, based on local available information from its neighbors:

$$\dot{z}_i(t) = u_i = \sum_{j \in \mathcal{N}_i} k_{ij} e^{\iota \alpha_{ij}} (z_j(t) - z_i(t)), \tag{3}$$

where $k_{ij} > 0$ is called *pursuit strength*, and $\alpha_{ij} \in [-\pi, \pi)$ is called *pursuit angle*. Moreover, we call $w_{ij} = k_{ij} e^{\iota \alpha_{ij}}$ the *pursuit weight* from agent i to agent j, and call $w_i = \sum_{j \in \mathcal{N}_i} w_{ij}$ the *pursuit degree* of agent i. The control strategy is very intuitive and can be easily implemented with an image-based control. For example, if a follower tries to keep the image of its neighbor in a certain location on its retina (or image plane of artificial vision system) rather than the center, then a pursuit angle results.

In what follows, we shall call the overall system under a pursuit strategy (3) a *pursuit system*, of which the leaders' trajectories are inputs and the followers' trajectories are outputs.

3.1. From DAG to SFG

Without loss of generality, suppose that the initial positions of all followers are zero. Then, applying the Laplace transform on both sides of (3), we have

$$sZ_i(s) = \sum_{j \in \mathcal{N}_i} w_{ij} (Z_j(s) - Z_i(s)) \tag{4}$$

or

$$Z_i(s) = \sum_{j \in \mathcal{N}_i} \frac{w_{ij}}{s + w_i} Z_j(s) \tag{5}$$

where $Z_i(s)$ and $Z_j(s)$ are the Laplace transforms of $z_i(t)$ and $z_j(t)$, respectively. Now we construct the corresponding SFG for the pursuit system from its associated DAG. We relabel each node i by its corresponding Laplace transform $Z_i(s)$ and place the gain (transfer function) $W_{ij} = \frac{w_{ij}}{s+w_i}$ on edge (j, i). Then the SFG for the pursuit system is obtained. An example is presented in Fig. 1.

In an SFG for a pursuit system, the source nodes correspond to the leaders, the sink and mixed nodes correspond to the followers, and the gains are related to the pursuit weights. In our setup, there might exist multiple sources and sinks, but there is no loop as the interaction topology is acyclic. For each pair of leaders and followers, we construct a single-source single-sink sub-SFG, which is the subgraph in the SFG containing all the paths from the leader to the follower. The leader is a single source and the follower is a single sink in the sub-SFG. Thus, each sub-SFG corresponds to a single-input single-output (SISO) linear time-invariant (LTI) system. An example of constructing a sub-SFG from a source to a non-source node is given in Fig. 2.

3.2. Stability analysis

We now come to study stability properties of pursuit systems. First, we introduce the concept of stability. In linear control theory, for an SISO LTI system with a strictly proper rational transfer function, there are two types of equivalent stability concepts (Dorato, Lepschy, & Viaro, 1994): (i) The transient component converges to 0. (ii) Every bounded input produces a bounded output, which is called BIBO stability. In the paper, the pursuit system is said to be BIBO stable if the transient component of every follower converges to 0. The concept is deduced from the BIBO stability definition for multi-variable systems in (Chen, 1984, chap. 8).

Next, we present our main result.

Theorem 3.1. *A pursuit system on a directed acyclic graph with each follower's dynamics defined in* (3) *is BIBO stable if and only if the pursuit degree of every follower has a positive real part, i.e,* $\mathrm{Re}(w_i) > 0$ *for every follower* i.

Proof. Consider any sub-SFG with a leader l as its source and a follower k as its sink. According to Mason's rule, the overall gain of the sub-SFG takes the form

$$T^l = \sum T_p^l = \sum_p \prod_q \hat{W}_{pq}^l \tag{6}$$

where T_p^l is the forward path gain of the p-th forward path from the leader l to the follower k and $\hat{W}_{pq}^l$ is the gain of the q-th edge on the p-th path in the sub-SFG. If this edge is labeled as (j, i) in the DAG, then $\hat{W}_{pq}^l = W_{ij} = \frac{w_{ij}}{s+w_i}$. Note that the poles of T^l are the collection of poles of $\hat{W}_{pq}^l$, namely, $-w_i$. Moreover, according to the superposition principle, the output $Z_k(s)$ is given by

$$Z_k(s) = \sum_l T^l R^l(s),$$

where $R^l(s)$ is the input of leader l. Now considering the impulse function for all leaders, we have

$$Z_k(s) = \sum_{l=1}^{m} T^l = \sum_{l=1}^{m} \sum_p \prod_q \hat{W}_{pq}^l.$$

Thus, it follows that the impulse response $z_k(t)$ converges to zero if and only if the pursuit degrees of agents in the paths from the leaders to the follower k all have positive real parts. Applying this argument to any follower in the group, we then obtain that the pursuit system is BIBO stable if and only if the pursuit degree w_i of every follower has positive real part. ■

Finally, let us consider pursuit systems in a 1D space, where each agent i's position z_i becomes a real number r_i. The dynamics of each follower becomes

$$\dot{r}_i(t) = \sum_{j\in\mathcal{N}_i} k_{ij}(r_j(t) - r_i(t)). \tag{7}$$

This is a degenerate case of pursuit system in a 2D space. Consequently, the following result can be easily obtained from Theorem 3.1.

Corollary 3.1. *A pursuit system on a directed acyclic graph with each follower's dynamics defined in (7) is always BIBO stable.*

3.3. Collective motions and formations

If a pursuit system is BIBO stable, all the transient components of the followers' trajectories converge to zero. In this subsection, we study the steady-state properties. In other words, we are interested in the type of collective motions and formations that result from the pursuit strategies. In particular, we consider three types of ordered motions for the leaders—stationary, uniform rectilinear motions, and uniform circular motions, to see how the leaders govern the entire group behaviors.

First we introduce the description of a formation for a group of N agents in the plane. A *formation* is defined by a set of N complex numbers $[c_1, c_2, \ldots, c_N]$. Such a description is independent of the choice of a complex plane. In other words, when we say a group of N agents are in a formation $[c_1, c_2, \ldots, c_N]$, the trajectories of the agents in an inertia frame can be written as $z_i(t) = e^{\iota\theta(t)}c_i + \xi(t)$, $(i = 1, \ldots, N)$ for some rotation $e^{\iota\theta(t)}$ and translation $\xi(t) \in \mathbb{C}$.

Note that for a pursuit system on a directed acyclic graph, the trajectory of each follower can be calculated in terms of its neighbors' steady-state trajectories. Hence, before presenting our main result on achievable formations, we first investigate the steady-state trajectories of each agent.

Lemma 3.1. *Consider a pursuit system with each follower's dynamics defined in (3). Suppose* $\mathrm{Re}(w_i) > 0$.

(i) *If $z_j(t) = c_j 1(t)$ for $j \in \mathcal{N}_i$ where $c_j \in \mathbb{C}$ is a constant, then the steady-state trajectory of agent i is*

$$\bar{z}_i(t) = \frac{\sum_{j\in\mathcal{N}_i} c_j w_{ij}}{w_i} 1(t). \tag{8}$$

(ii) *If $z_j(t) = ct$ for $j \in \mathcal{N}_i$ where $c \in \mathbb{C}$ is a constant, then the steady-state trajectory of agent i is*

$$\bar{z}_i(t) = ct - \frac{c}{w_i} 1(t). \tag{9}$$

(iii) *If $z_j(t) = \rho_j e^{\iota(\gamma t+\psi_j)}$ for $j \in \mathcal{N}_i$ where $\rho_j \in \mathbb{R}^+$, γ, $\psi_j \in \mathbb{R}$ are constants, then the steady-state trajectory of agent i is*

$$\bar{z}_i(t) = \frac{e^{\iota\gamma t}}{\iota\gamma + w_i} \sum_{j\in\mathcal{N}_i} w_{ij}\rho_j e^{\iota\psi_j}. \tag{10}$$

Proof. We prove (ii). The other two can be proved similarly. For $j \in \mathcal{N}_i$, $z_j(t) = ct$. Then its Laplace transform is $Z_j(s) = \frac{c}{s^2}$. From (5), we have

$$\begin{aligned} Z_i(s) &= \sum_{j\in\mathcal{N}_i} \frac{c w_{ij}}{s^2(s + w_i)} \\ &= \sum_{j\in\mathcal{N}_i} \frac{c w_{ij}}{w_i} \left(\frac{\frac{-1}{w_i}}{s} + \frac{1}{s^2} + \frac{\frac{1}{w_i}}{s + w_i} \right) \\ &= \frac{-\frac{c}{w_i}}{s} + \frac{c}{s^2} + \frac{\frac{c}{w_i}}{s + w_i}. \end{aligned} \tag{11}$$

Applying the inverse Laplace transform to (11), we obtain

$$z_i(t) = ct - \frac{c}{w_i} 1(t) + \frac{c}{w_i} e^{-w_i t}.$$

Since $\mathrm{Re}(w_i) > 0$ by our assumption, the steady-state trajectory of agent i turns out to be $\bar{z}_i(t)$ in (9). ■

Now we use Lemma 3.1 repeatedly to obtain our main result on formations.

Theorem 3.2. *Consider a pursuit system on a directed acyclic graph with each follower's dynamics defined in (3). Suppose the pursuit system is BIBO stable.*

(i) *If the leaders are in a static formation $[c_1, c_2, \ldots, c_{N_l}]$, then the group of agents achieves a formation*

$[c_1, c_2, \ldots, c_{N_l}, c_{N_l+1}, \ldots, c_N]$

where $c_i = \frac{\sum_{j\in\mathcal{N}_i} c_j w_{ij}}{w_i}$ for $i = N_l + 1, \ldots, N$.

(ii) *If the leaders perform uniform rectilinear motions (i.e., $z_i(t) = c_i + ct$ for $i = 1, \ldots, N_l$), then the group of agents perform uniform rectilinear motions with the same velocity c while achieving a formation*

$[c_1, c_2, \ldots, c_{N_l}, c_{N_l+1}, \ldots, c_N]$

where $c_i = \frac{-c+\sum_{j\in\mathcal{N}_i} c_j w_{ij}}{w_i}$ for $i = N_l + 1, \ldots, N$.

(iii) *If the leaders perform uniform circular motions (i.e., $z_i(t) = c_0 + \rho_i e^{\iota(\gamma t+\psi_i)}$ for $i = 1, \ldots, N_l$), then the group of agents perform uniform circular motions with the same center c_0 and the same frequency γ while achieving a formation*

$[c_1, c_2, \ldots, c_{N_l}, c_{N_l+1}, \ldots, c_N]$

where $c_i = \rho_i e^{\iota\psi_i}$ for $i = 1, 2, \ldots, N_l$ and $c_i = \frac{\sum_{j\in\mathcal{N}_i} w_{ij} c_j}{\iota\gamma + w_i}$ for $i = N_l + 1, \ldots, N$.

The proof is in Appendix.

Remark 3.1. When leaders' motion is a linear combination of a series of rectilinear motions and circular motions, the collective motions and formations of the group can still be obtained from Theorem 3.2 according to the superposition principle.

Remark 3.2. Theorem 3.2 can be used to synthesize control parameters (namely, gains and offset angles) in order to attain a desired formation, which could be stationary or moving. Compared to existing work (e.g., (Lin et al., 2004; Ren, 2007a,b)) on formation control based on consensus algorithms, our approach does not require a common sense of direction for all agents. Moreover, our approach uses less information from neighbors (only relative positions) in contrast to the formation control strategies in Wang (1989), Shi et al. (2006), Shi et al. (2007), Bai et al. (2008) and Sumpter, Buhl, Biro, and Couzin (2008) where the agents need to know the velocities of neighbors and/or leaders.

4. Pursuit systems (Double-integrator dynamics)

The single-integrator model is suitable for analyzing collective behaviors in nature due to its simplicity. However, results obtained for single-integrator kinematics may not be applicable to real mobile robots because mobile robots have more complex dynamics. Since many dynamics of mobile robots can be transformed into double-integrator dynamics through feedback linearization (Lawton, Beard, Young, Syst, & Tucson, 2003), we also study double-integrator models

$$\ddot{z}_i = u_i, \tag{12}$$

where $z_i \in \mathbb{C}$ is the position of agent i in the plane and u_i is the control input.

Suppose the neighboring relationship of a network of N agents is described by a directed acyclic graph. We consider the following pursuit strategy for follower i

$$\ddot{z}_i(t) = \sum_{j\in\mathcal{N}_i} k_{ij} e^{\iota\alpha_{ij}} (z_j(t) - z_i(t)) - \eta_i \dot{z}_i(t), \tag{13}$$

where $\eta_i > 0$ is the damping gain and the other control parameters are the same as in (3). The damping gain is used to stabilize the formation.

Without loss of generality, we suppose that the initial position and velocity of each follower are zero. Applying the Laplace transform on both sides of (13), we get

$$Z_i(s) = \sum_{j\in\mathcal{N}_i} \frac{w_{ij} Z_j(s)}{s^2 + \eta_i s + w_i}. \tag{14}$$

Thus, the gain on edge (j, i) in the associated SFG for the pursuit system is

$$W_{ij} = \frac{w_{ij}}{s^2 + \eta_i s + w_i}.$$

4.1. Stability analysis

First, we present a stability result for the pursuit system with double-integrator dynamics.

Theorem 4.1. *A pursuit system on a directed acyclic graph with each follower's dynamics defined in (13) is BIBO stable if and only if* $\frac{\mathrm{Re}(w_i)}{(\mathrm{Im}(w_i))^2} > \frac{1}{\eta_i^2}$ *for every follower i.*

Proof. Similar to the proof of Theorem 3.1, we know that the pursuit system is BIBO stable if and only if the roots of the complex polynomial equation $s^2 + \eta_i s + w_i = 0$ are in the open left half complex plane for every follower i. According to Chen and Tsai (1993), the roots are in the open left half complex plane if and only if $\frac{\mathrm{Re}(w_i)}{(\mathrm{Im}(w_i))^2} > \frac{1}{\eta_i^2}$. Hence, the conclusion follows. ■

When a group of agents are in a 1D space, agent i's position z_i is a real number, denoted by r_i. The dynamics of each follower i then become

$$\ddot{r}_i(t) = \sum_{j\in\mathcal{N}_i} k_{ij}(r_j(t) - r_i(t)) - \eta_i \dot{r}_i(t). \tag{15}$$

The pursuit degree of agent i turns out to be $k_i = \sum_{j\in\mathcal{N}_i} k_{ij} > 0$. Thus, the following result is obtained immediately from Theorem 4.1.

Corollary 4.1. *A pursuit system on a directed acyclic graph with each follower's dynamics defined in (15) is always BIBO stable.*

4.2. Collective motions and formations

Next, we present a result on collective motions and formations of the pursuit system with double-integrator dynamics. This result is similar to the one for the pursuit system with single-integrator kinematics. So the proof is omitted due to space limitations.

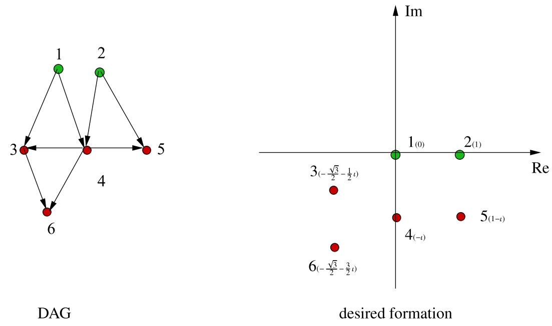

Fig. 3. A DAG and a desired formation (the complex in bracket is c_i).

Theorem 4.2. *Consider a pursuit system on a directed acyclic graph with each follower's dynamics defined in (13). Suppose the pursuit system is BIBO stable.*

(i) *If the leaders are in a static formation* $[c_1, c_2, \ldots, c_{N_l}]$, *then the group of agents achieves a formation*

$$[c_1, c_2, \ldots, c_{N_l}, c_{N_l+1}, \ldots, c_N]$$

where $c_i = \frac{\sum_{j\in\mathcal{N}_i} c_j w_{ij}}{w_i}$ *for* $i = N_l + 1, \ldots, N$.

(ii) *If the leaders perform uniform rectilinear motions (i.e.,* $z_i(t) = c_i + ct$ *for* $i = 1, \ldots, N_l$*), then the group of agents perform uniform rectilinear motions with the same velocity c while achieving a formation*

$$[c_1, c_2, \ldots, c_{N_l}, c_{N_l+1}, \ldots, c_N]$$

where $c_i = \frac{-c\eta_i + \sum_{j\in\mathcal{N}_i} c_j w_{ij}}{w_i}$ *for* $i = N_l + 1, \ldots, N$.

(iii) *If the leaders perform uniform circular motions (i.e.,* $z_i(t) = c_0 + \rho_i e^{\iota(\gamma t + \psi_i)}$ *for* $i = 1, \ldots, N_l$*), then the group of agents performs uniform circular motions with the same center* c_0 *and the same frequency* γ *while achieving a formation*

$$[c_1, c_2, \ldots, c_{N_l}, c_{N_l+1}, \ldots, c_N]$$

where $c_i = \rho_i e^{\iota\psi_i}$ *for* $i = 1, 2, \ldots, N_l$ *and* $c_i = \frac{\sum_{j\in\mathcal{N}_i} w_{ij} c_j}{-\gamma^2 + \iota\gamma\eta_i + w_i}$ *for* $i = N_l + 1, \ldots, N$.

5. Simulation results

In this section, we present a simulation of forming a static formation in the plane and three simulations of achieving collective motions while holding some formation patterns in a 3D space. Single-integrator models are used for the simulations.

First, we show how the control parameters (gains k_{ij} and offset angles α_{ij}) are designed for a group of agents to attain a desired geometric formation given a directed acyclic graph describing their interaction relationship. Consider an example of six agents with two leaders labeled 1 and 2 and four followers labeled from 3 to 6. The directed acyclic graph is given in Fig. 3 (left). Two leaders keep stationary in a formation $[c_1, c_2]$ where $c_1 = 0$ and $c_2 = 1$ with respect to an inertial frame. The leaders can be beacons or landmarks in practice, governing the followers to reach a formation. The desired formation for the group including the leaders is $[c_1, c_2, c_3, \ldots, c_6]$, which is shown in Fig. 3 (right). We now present how to apply our results to synthesize control parameters to achieve the desired formation. First, we design w_{41} and w_{42}. According to Theorem 3.2(i), we have

$$-\iota = c_4 = \frac{c_1 w_{41} + c_2 w_{42}}{w_{41} + w_{42}} = \frac{w_{42}}{w_{41} + w_{42}}.$$

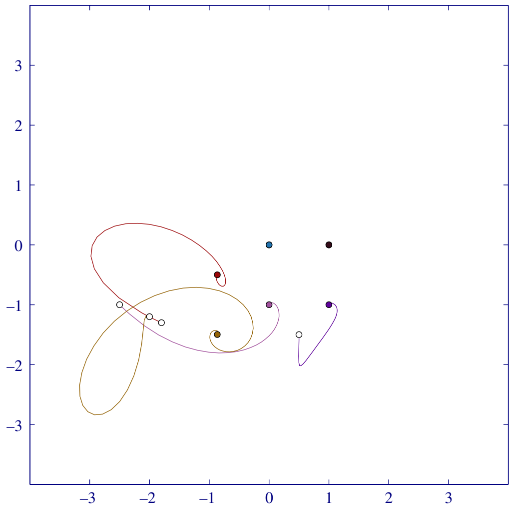

Fig. 4. Achieving a static formation.

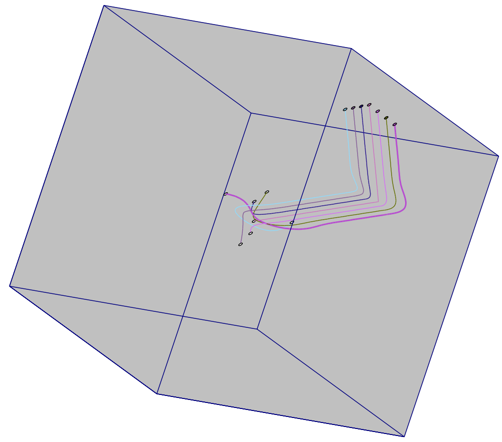

Fig. 5. V-formation.

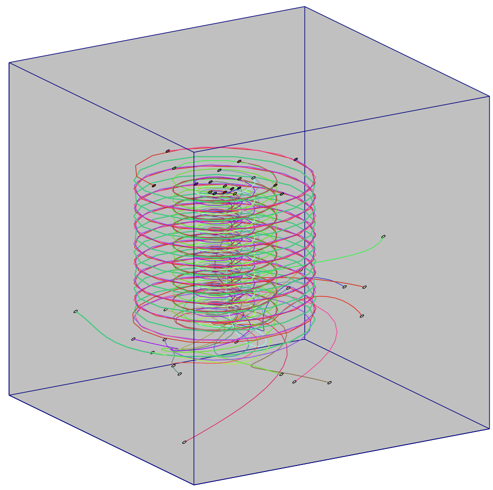

Fig. 6. Vortex motion.

Also, the parameters should satisfy $\mathrm{Re}(w_{41}+w_{42}) > 0$ for stability. Thus, we select $w_{41} = 1$ and then we get $w_{42} = \frac{\sqrt{2}}{2}e^{-\iota\frac{3\pi}{4}}$. From the pursuit weight, we can obtain the pursuit strength k_{ij} and the pursuit angle α_{ij}. Taking w_{42} as an example, the pursuit strength and the pursuit angle are $k_{42} = \frac{\sqrt{2}}{2}$ and $\alpha_{42} = -\frac{3\pi}{4}$, respectively. Similarly, we select $w_{31} = w_{63} = e^{\iota\frac{2\pi}{3}}$, $w_{34} = w_{64} = 1$, $w_{52} = 1$, and $w_{54} = \iota$. Then under the control law (3) with parameters selected as above, the group of agents eventually forms the desired formation for any initial state. Simulated trajectories with arbitrary initial positions are shown in Fig. 4.

Next, we present a few simulations in a 3D space using our control strategy to mimic emergent phenomena observed in nature. Before presenting our simulations, we give some conclusions of pursuit systems in 3D space based on the results in a 2D space and a 1D space.

Let q_i denote the position of agent i in $\mathbb{R}^3$. Rotating a vector in 3D-space depends on both the rotation angle and the rotation axis. Specifically, in the Cartesian coordinate system XYZ, we consider a rotation matrix $C(\alpha)$ of an angle α with respect to the axis of $[0, 0, 1]^T$ (Z-axis). That is,

$$C(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The supposition is compatible with the observation that birds can sense the force of gravity and choose the Z-axis accordingly. For a pursuit system in 3D space, if follower i pursues its neighbors according to the following equation

$$\dot{q}_i = \sum_{j\in\mathcal{N}_i} k_{ij} C(\alpha_{ij})(q_j - q_i), \qquad (16)$$

it can be decomposed into two uncoupled systems (with one in a 2D space and the other in a 1D space):

$$\dot{z}_i = \sum_{j\in\mathcal{N}_i} k_{ij} e^{\iota\alpha_{ij}}(z_j - z_i) \qquad (17)$$

and

$$\dot{r}_i = \sum_{j\in\mathcal{N}_i} k_{ij}(r_j - r_i) \qquad (18)$$

where $z_i = q_{ix} + \iota q_{iy} \in \mathbb{C}$ corresponds to the first two components of q_i, and r_i is the third component of q_i.

First, we simulate a behavior of V-formation flying in 3D space using our pursuit strategy, which is shown in Fig. 5. The result in the simulation is assured by Theorem 3.2(ii). Next, we consider that leaders perform uniform circular motions in XY-plane and perform uniform rectilinear motions in Z-direction. The group of agents thus attains and maintains a formation while following the motions of leaders. The simulated collective behavior is shown in Fig. 6, which looks similar to the vortex motion observed in nature. More complicatedly, if the motion of leaders in the XY-plane is a combination of uniform circular motions and uniform rectilinear motions, and in the Z-direction they perform uniform rectilinear motions, then the entire group achieves a formation in moving which is similar to the tornado motion as in Fig. 7.

6. Remarks and conclusions

In this paper, we investigate collective motions and formations of pursuit systems on DAGs for both single-integrator kinematics and double-integrator dynamics. In a pursuit system, each follower pursues its neighbors with a certain pursuit strength and pursuit angle. Based on the signal flow graph and Mason's rule, necessary and sufficient conditions are derived for BIBO stability. Moreover, achievable collective motions and formations are analyzed. Simulations are provided to show how the control parameters are synthesized to attain any desired formation. Our results are not only

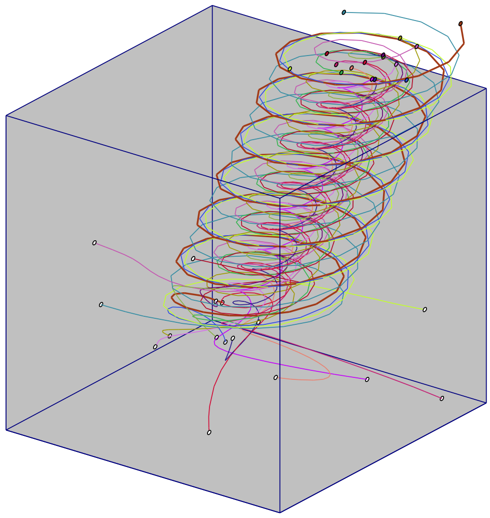

Fig. 7. Tornado motion.

meaningful for achieving stable swarm behaviors, but also useful for formation control in engineering applications, as the control law is simple enough to be implemented with image-based control. The approach is within the decentralized control architecture requiring that controls are computed and locally applied by an agent using information sensed by that agent, while the control design has been centrally computed for the purpose of formation control. However, there is the possibility of requiring decentralization not only for control law implementation, but also for control law design, especially adaptive design with limited or no intervention from a central entity. Another important issue in swarming and formation control is collision avoidance, which has not been addressed in the paper. Nevertheless, the offset angle can be used to shift an agent's position away from its proximate neighbors if we can dynamically adjust the offset angles. It would be an interesting future work to exploit the rules of avoiding collisions by dynamically changing the offset angle. In addition to collision avoidance, dynamic pursuit strengths and pursuit angles can also be utilized to maintain the connectivity of the group when each agent has only a limited field of view with its onboard sensor. This also remains an open problem.

Appendix

Proof of Theorem 3.2. (i) For a directed acyclic graph, there exists a subset of followers whose neighbors are only leaders. We denote this set by $\mathcal{A}_1$. Then from Lemma 3.1, it follows that for any agent $i \in \mathcal{A}_1$, its steady state is given as

$$\bar{z}_i(t) = \frac{\sum\limits_{j\in\mathcal{N}_i} c_j w_{ij}}{w_i} \tag{19}$$

with all $c_j, j \in \mathcal{N}_i$ known. Next, we can find a subset of agents denoted by $\mathcal{A}_2$ whose neighbors are leaders and/or agents in $\mathcal{A}_1$. Thus, we can calculate their steady states with (19) because all the neighbors' steady states are known now. In this way, we can obtain all the agents' steady states and know that the group of agents achieves a formation as stated in (i).

(ii) As the leaders perform uniform rectilinear motions (i.e., $z_i(t) = c_i + ct$ for $i = 1, \ldots, N_l$), according to the superposition principle and Lemma 3.1, it is known that the steady-state trajectory of any agent i in the subset $\mathcal{A}_1$ can be calculated in terms of the formula

$$\bar{z}_i(t) = ct + \frac{-c + \sum\limits_{j\in\mathcal{N}_i} c_j w_{ij}}{w_i}. \tag{20}$$

With the same argument as for (i), the steady-state trajectory of each agent can be calculated with (20). Thus, according to the definition of a formation, the group of agents achieves a formation $[c_1, c_2, \ldots, c_N]$ up to a translation ct. The translation ct means that the entire group performs uniform rectilinear motions.

(iii) As the leaders perform uniform circular motions (i.e., $z_i(t) = c_0 + \rho_i e^{\iota(\gamma t+\psi_i)}$ for $i = 1, \ldots, N_l$), according to the superposition principle and Lemma 3.1, it is known that the steady-state trajectory of any agent i in the subset $\mathcal{A}_1$ can be calculated in terms of the formula

$$\bar{z}_i(t) = c_0 + e^{\iota\gamma t}\left(\frac{\sum\limits_{j\in\mathcal{N}_i} w_{ij}\rho_j e^{\iota\psi_j}}{\iota\gamma + w_i}\right). \tag{21}$$

Repeatedly, the steady-state trajectory of each agent can be calculated with (21). Thus, according to the definition of a formation, the group of agents achieves a formation $[c_1, c_2, \ldots, c_N]$ as stated in (iii) up to a translation c_0 and a rotation $e^{\iota\gamma t}$. It means that the entire group also performs uniform circular motions with the same center c_0. ■

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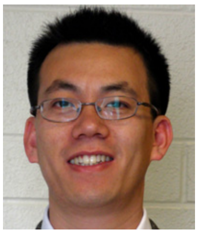
Wei Ding received the B.S. degree in electrical engineering from Zhejiang University, China in 2006. He is currently working towards the Ph.D. degree in control theory and control engineering at Zhejiang University, China. His current research interest is the cooperative control of multi-agent systems, including consensus, flocking, and formation control.

Gangfeng Yan received the B.S. and M.S. degrees in Control Theory and Control Engineering from Zhejiang University, China, in 1981 and 1984, respectively. He is currently a professor at the Department of Systems Science and Engineering, Zhejiang University. His research interests include hybrid systems, neural networks, and cooperative control.



Zhiyun Lin received his Ph.D. degree in electrical and computer engineering from the University of Toronto, Canada, in 2005. From 2005 to 2007, he was a postdoctoral researcher at the Department of Electrical and Computer Engineering at the University of Toronto, Canada. In 2007 he joined the Department of Systems Science and Engineering at Zhejiang University, China, as a Research Professor. His recent research is concentrated on large scale networked multi-agent systems, cooperative control of distributed robot systems, wireless sensor networks, switched and hybrid systems, and locomotion control of legged robots. He has authored or co-authored one monograph and about 30 refereed publications.