Ergodic of Harris Recurrent Markov Chain

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1 INTRODUCTION

1.1 WHAT IS ERGODIC?

Ergodic theory concerned the long time behavior of the object that we are interested in. It is a fundamental problem in Dynamical System and has its counter part in study of Stochastic Process. Here we formulate the Ergodicity in these two area.

1.1.1 ERGODIC OF DYNAMICAL SYSTEM.

Let $T : X \rightarrow X$ be a measure-preserving transformation on a measure space $(X, \Sigma, \mu)$ and $f$ be a $\mu$-integrable function, i.e., $f \in L^1(\mu)$.

**Definition 1.1.** Starting from some initial point $x$, the time average of $f$ over iterations of $T$ is defined as the average (if it exists):

$$\hat{f}(x) = \lim_{x \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^{k-1}(x))$$

**Definition 1.2.** If $\mu(X)$ is finite and nonzero, we defined space average of $f$ as:

$$\bar{f} = \frac{1}{\mu(X)} \int f \, d\mu$$

Under some condition, the dynamical system has Ergodicity:

$$\lim_{x \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^{k-1}(x)) = \frac{1}{\mu(X)} \int f \, d\mu$$

Thus, if we has a ergodic system, the time average asymptotically equals space average. This result can be paraphrased as Law of Large Number.

1.1.2 ERGODIC OF STOCHASTIC PROCESS.

Let $\{X_n\}_{n \geq 0}$ be a sequence of random variable on probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

**Definition 1.3.** The Empirical Mean of $\{X_n\}_{n \geq 0}$ is defined as:

$$E_{\mu}f(X) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(X_k)$$

**Definition 1.4.** The Expectation of $\{X_n\}_{n \geq 0}$ is defined as:

$$E_{\mu}f(X) = \int f \, d\mu$$

Under some condition, the empirical mean of stochastic process asymptotically equals expectation, which is the famous Law of Large Number:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(X_k) = \int f \, d\mu$$
Therefore, we find a correspondence between **Time average** with **Empirical Mean**, **Space average** with **Expectation** and **Ergodicity** with **Law of Large Number**.

<table>
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1.2 **Problem and Approach of this report.**

In this report, we pursue ergodic of the discrete time Markov Chain with continuous state. Ergodic of Markov Chain plays a fundamental role in the methodology of Monte-Carlo integration. We explore the condition which establish the ergodic theorem of the discrete time Markov Chain with continuous state and wish the exploration would make us obtain a better understanding the nature of Markov Chain and Monte-Carlo integration.

2 **Pursue Ergodic**

2.1 **Ergodic of Regenerative Sequence**

In this subsection, we gives the definition of regenerative sequence of random variable and proof the Ergodic Theorem of regenerative sequence.

2.1.1 **Define regenerative sequence**

**Definition 2.1.** A sequence of random variable \( \{X_n\}_{n \geq 0} \) defined on probability space \((\Omega, \mathcal{F}, P)\) with values in a measurable space \((S, \mathcal{S})\) is called regenerative if there exists a sequence of random times \(0 < T_1 < T_2 < \cdots < T_n < \cdots\) such that the excursions

\[
\eta_j \equiv (X_{T_j}, X_{T_j+1}, \cdots, X_{T_{j+1}}, T_{j+1} - T_j), j \leq 1
\]

are independent and identically distributed (i.i.d).

**Why Regenerate Sequence?** We can show that

- In discrete time countable state case, irreducible recurrent Markov Chain is regenerative;
- In discrete time continuous state case, Harris recurrent Markov Chain satisfied minimization condition is regenerative.

In terms of regenerate sequence, we can discuss Markov Chain with countable and continuous state case at the same time.

2.1.2 **Ergodic Theorem of regenerative sequence**

Here, we establish the **Ergodic Theorem of Regenerative Sequence**.

**Theorem 1.** Given a regenerative sequence \( \{X_n\}_{n \geq 0} \) with regeneration times \( \{T_i\}_{i \geq 1} \). Let \( \tilde{\pi}(A) \equiv E(\sum_{j=T_1}^{T_2} I_A(X_j)) \) for \( A \in \mathcal{F} \) and suppose \( \tilde{\pi}(S) \equiv E(T_2 - T_1) < \infty \). Define \( \pi(\cdot) \equiv \tilde{\pi}(\cdot)/\tilde{\pi}(S) \). Then

\[
\frac{1}{n} \sum_{j=0}^{n} f(X_j) \to \int f d\pi
\]

with probability 1 for any \( f \in L^1(S, \mathcal{S}, \pi) \).
Proof. Since \( f \) is \( L_1 \), it suffices to consider nonnegative \( f \).
Let \( N_n \) denote the number of excursions that \( X_n \) in, i.e., \( N_n = k \) if \( T_k \leq n < T_{k+1} \). Let
\[
Y_0 = \sum_{j=0}^{T_{i+1}-1} f(X_j), Y_i = \sum_{j=T_i}^{T_{i+1}-1} f(X_j), i \geq 1
\]
then
\[
Y_0 + \sum_{i=1}^{N_n-1} Y_i \leq n \sum_{i=0}^{N_n} f(X_i) \leq Y_0 + \sum_{i=1}^{N_n} Y_i
\]
Since \( \{Y_i\} \) are i.i.d., use SLLN, we have
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n} f(X_i) = \frac{EY_1}{E(T_2 - T_1)} = \int f d\pi = \int f d\pi
\]
\( \square \)

2.2 Ergodic of Harris Recurrent Markov Chain

In this subsection, we define Harris recurrent Markov Chain and minimization condition. The Fundamental Regenerative Theorem state the Harris recurrent Markov Chain satisfied minimization condition is a regenerative sequence. From the ergodic of regenerative sequence, we can obtain the ergodic of Harris recurrent Markov Chain.

2.2.1 Definition of Harris Irreducible and Harris Recurrent Markov Chain

Definition 2.2. A function \( P : \mathbb{S} \times \mathcal{A} \to [0, 1] \) is called a transition kernel on \( \mathbb{S} \) if
1. for all \( x \in \mathbb{S} \), \( P(x, \cdot) \) is a probability measure on \( (\mathbb{S}, \mathcal{A}) \).
2. for all \( A \in \mathcal{A} \), \( P(\cdot, A) \) is an \( \mathcal{A} \)-measurable function from \( \mathbb{S} \times \mathcal{A} \to [0, 1] \).

Definition 2.3. A sequence of random variable \( \{X_n\}_{n \geq 0} \) defined on probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) with values in a measurable space \( (\mathbb{S}, \mathcal{A}) \) is called a Markov Chain with transition kernel \( P(\cdot, \cdot) \) if for all \( A \in \mathcal{A} \),
\[
P(X_{n+1} \in A|\mathcal{F}_n) = P(X_{n+1} \in A|\sigma(X_n))
\]
with probability 1, for all \( n \leq 0 \), for any initial distribution \( X_0 \), where \( \mathcal{F}_n \equiv \sigma(\{X_j: 0 \leq j \leq n\}) \) is the sub-\( \sigma \)-algebra of \( \mathcal{A} \) generated by \( X_n \).

Definition 2.4. Let \( \phi \) be a nonzero \( \sigma \)-finite measure on \( (\mathbb{S}, \mathcal{A}) \). The Markov chain \( \{X_n\}_{n \geq 0} \) is said to be Harris-\( \phi \)-irreducible if for any set \( A \in \mathcal{A} \) with non-zero \( \phi \)-measure, starting from any point \( x \in \mathbb{S} \), the first entrance time to \( A \) is finite with non-zero probability, i.e., for any \( A \in \mathcal{A} \),
\[
\phi(A) > 0 \Rightarrow L(x, A) \equiv P_x(T_A < \infty) > 0
\]
for all \( x \in \mathbb{S} \), where
\[
T_A = \begin{cases} 
\min\{n : n \geq 1, X_n \in A\} & \text{if } X_n \notin A, \forall n.
\end{cases}
\]
More, \( \{X_n\}_{n \geq 0} \) is said to be Harris-\( \phi \)-recurrent if it satisfied
\[
\phi(A) > 0 \Rightarrow L(x, A) \equiv P_x(T_A < \infty) = 1
\]
for any \( A \in \mathcal{A} \), for all \( x \in \mathbb{S} \).
2.2.2 Minimization Condition

**Definition 2.5.** A Markov Chain \( \{X_n\}_{n \geq 0} \) with state space \( (\mathcal{S}, \mathcal{F}) \) and transition kernel \( P(\cdot, \cdot) \) is said to satisfy the **minimization condition** \((A_0, \alpha, n_0, \nu)\) if there exists a set \( A_0 \in \mathcal{F}, 0 < \alpha < 1, n_0 \geq 1, \) a probability measure \( \nu(\cdot) \) on \( (\mathcal{S}, \mathcal{F}) \) such that for all \( x \in A_0, \)

\[
P^{n_0}(x, A) \geq \alpha \nu(A)
\]

for all \( A \in \mathcal{F}. \)

**Why Minimization Condition?** Minimization condition is part of sufficient condition that make a Markov Chain be a regenerate sequence. If we establish regenerate property of Markov Chain, we can use ergodic of regenerate sequence to achieve ergodic of Markov Chain.

A sufficient condition that make a Markov Chain satisfied the minimization condition is **Harris irreducibility**.

**Theorem 2.** Given a Harris \( \phi \)-irreducible Markov Chain \( \{X_n\}_{n \geq 0} \) with countably generated state space \( (\mathcal{S}, \mathcal{F}) \). For every \( B_0 \in \mathcal{F} \) with \( \phi(B_0) > 0 \), there exists a set \( A_0 \subset B_0 \) s.t. \( \{X_n\}_{n \geq 0} \) satisfy the minimization condition \((A_0, \alpha, n_0, \nu)\).

2.2.3 Fundamental Regenerative Theorem

Now, if, starting from any point in state space, the 1st entrance time of set \( A_0 \) in minimization condition is finite with probability one, we can establish the Markov Chain a regenerate sequence, which is the consequence of **Fundamental regenerative Theorem**:  

**Theorem 3.** Given a Markov Chain \( \{X_n\}_{n \geq 0} \) with state space \( (\mathcal{S}, \mathcal{F}) \) satisfied the minimization condition \((A_0, \alpha, n_0, \nu)\). If for all \( x \in \mathcal{S}, \)

\[
P_x(T_{A_0} < \infty) = 1
\]

then \( \{X_n\}_{n \geq 0} \) is regenerative.

Thus, with definition of **Harris \( \phi \)-recurrence**, we have:

**Theorem 4.** A Harris \( \phi \)-recurrent Markov Chain \( \{X_n\}_{n \geq 0} \) with countably generated state space \( (\mathcal{S}, \mathcal{F}) \) is regenerative.

Until now, we learn that

**Harris \( \phi \)-recurrence establishes the Markov Chain a regenerate sequence.**

2.2.4 Ergodic Theorem of Harris Recurrent Markov Chain

To achieve the ergodic of Harris \( \phi \)-recurrent Markov Chain, the remaining part of sufficient condition is the **existence of stationary probability measure**.

**Definition 2.6.** A probability measure \( \pi \) on \( (\mathcal{S}, \mathcal{F}) \) is called **stationary** for a transition kernel \( P(\cdot, \cdot) \) if for all \( A \in \mathcal{F}, \)

\[
\pi(A) = \int_{\mathcal{S}} P(x, A) \pi(dx)
\]
The existence of stationary probability measure of a Harris recurrent Markov Chain is given by:

**Theorem 5.** A necessary and sufficient condition for the existence of a stationary distribution for a Harris recurrent Markov Chain \( \{X_n\}_{n \geq 0} \) is that: \( \{X_n\}_{n \geq 0} \) satisfied the minimization condition \((A_0, \alpha, n_0, \nu)\) and

\[
E_\nu T_{A_0} < \infty
\]

Finally, we establish the **Ergodic Theorem of Harris \( \phi \)-recurrent Markov Chain.**

**Theorem 6.** Given a Harris \( \phi \)-recurrent Markov Chain \( \{X_n\}_{n \geq 0} \) with countably generated state space \((\mathcal{S}, \mathcal{F})\). If there exists a stationary probability measure \( \pi \), then for any \( f \in L^1(\mathcal{S}, \mathcal{F}, \pi) \),

\[
\frac{1}{n} \sum_{j=0}^{n-1} f(X_j) \to \int f \, d\pi
\]

with probability 1 \( (P_x) \).

### 3 Conclusion

- Harris Irreducible
- Harris Recurrent
- Minimization Condition
- Existence of Recurrent Small Set
- Existence of Stationary Measure
- Regenerative Sequence
- Ergodicity

### References
