Coasian Equilibria in Sequential Auctions

Qingmin Liu* Konrad Mierendorff†
Xianwen Shi‡

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Abstract

We study stationary equilibria in a sequential auction setting. A seller runs a sequence of standard auctions to sell an indivisible object to potential buyers. The seller can commit to the rule of the auction and the reserve price of current period, but not to reserve prices of future periods. We prove the existence of weak Markov equilibria and establish a uniform Coasian conjecture—at any point in time, in any weak Markov equilibrium, the seller's profit from running sequential auctions converges to the profit of running an efficient auction as the period length goes to zero.

1 Introduction

Consider the standard auction setting with a seller, a single indivisible object, and multiple buyers whose values are drawn independently from a common distribution. Different from the classic auction model, if the object is not sold on previous occasions, the seller can sell it again with no predetermined deadline. In each time period

*Columbia University, qingmin.liu@columbia.edu
†University of College London, k.mierendorff@ucl.ac.uk
‡University of Toronto, xianwen.shi@utoronto.ca

until the object is sold, the seller posts a reserve price and holds a standard auction (e.g., second-price auction or first-price auction). Each buyer can either wait for a future auction or submit a bid no smaller than the reserve price. Waiting is costly—both the buyers and the seller discount at the same rate. Within a period, the seller is committed to the rules of the auction and the announced reserve price. The seller cannot, however, commit to future reserve prices. The seller's commitment power varies with the period length (or effectively with the discount factor). As the period length shrinks, the seller's commitment power diminishes. We are interested in the existence of weak Markov equilibria and the seller's equililibrium payoff in these equilibria in the continuous-time limit at which the seller's commitment power vanishes.

We prove the existence of weak Markov equilibria and establish a uniform Coasian conjecture—at any point in time, in any weak Markov equilibrium, the seller's profit from running sequential auctions converges to the revenue of running an efficient auction as the period length goes to zero.

The Coasian bargaining model with a single buyer is a special case of our setup. Coase (1972) argues that a price-setting monopolist completely loses her monopoly power and prices drop quickly to her marginal cost if she can revise prices frequently. Fudenberg et al. (1985) and Gul et al. (1986) confirm that every stationary equilibrium—stationary in the sense that the buyer's equilibrium strategy can only condition on the current price offer—satisfies the Coase conjecture. Ausubel and Deneckere (1989) show that, if there is "no gap" between the seller's reservation value and minimum valuation of the buyer, there is a continuum of non-stationary "reputational equilibria" in addition to the stationary Coasian equilibria. In contrast, if there is a "gap" so that the seller's reservation value is strictly below the lowest buyer valuation, as is the case in Fudenberg et al. (1985), the game is essentially a game with a finite horizon. All equilibria are stationary.

We focus on the no-gap case and our contribution is to extend the analysis of Coasian equilibria in Ausubel and Deneckere (1989) to the auction setting. For the

analysis of reputational equilibrium in the auction setting, see Liu et al. (2018).

2 Model

We consider the standard auction environment where a seller (she) wants to sell an indivisible object to n potential buyers (he). Buyer i privately observes his own valuation for the object $v^i \in [0,1]$. We use $(v^i, v^{-i}) \in [0,1]^n$ to denote the vector of the n buyers' valuations, and $v \in [0,1]$ to denote a generic buyer's valuation. Each v^i is drawn independently from a common distribution with full support, c.d.f. $F(\cdot)$, and a continuously differentiable density $f(\cdot)$ such that $f(v) > 0$ for all $v \in (0,1)$. The highest order statistic of the n valuations (v^i, v^{-i}) is denoted by $v^{(n)}$, its c.d.f. by $F^{(n)}$, and the density by $f^{(n)}$. The seller's reservation value for the object is constant over time and we normalize it to zero.

Time is discrete and the period length is denoted by Δ. In each period $t = 0, \Delta, 2\Delta, \ldots$, the seller runs a second-price auction (SPA) with a reserve price. To simplify notation, we often do not explicitly specify the dependence of the game on Δ. The timing within period t is as follows. First, the seller publicly announces a reserve price p_t for the auction run in period t, and invites all buyers to submit a valid bid, which is restricted to the interval $[p_t, 1]$. After observing p_t, all buyers decide simultaneously either to bid or to wait. If at least one valid bid is submitted, the winner and the payment are determined according to the rules of the second-price auction and the game ends. If no valid bid is submitted, the game proceeds to the next period. Both the seller and the buyers are risk-neutral and have a common discount rate $r > 0$. This implies a discount factor per period equal to $\delta = e^{-r\Delta} < 1$. If buyer i wins in period t and has to make a payment π^i, then his payoff is $e^{-rt}(v^i - \pi^i)$, and the seller's payoff is $e^{-rt}\pi^i$.

We assume that the seller has limited commitment power. She can commit to the reserve price that she announces for the current period: if a valid bid is placed, then the object is sold according to the rules of the announced auction and she cannot

renege. She cannot commit, however, to future reserve prices: if the object was not sold in a period, the seller can always run another auction with a new reserve price in the next period. She cannot promise to stop auctioning an unsold object, or commit to a predetermined sequence of reserve prices.

We denote by $h_t = (p_0, p_\Delta, \ldots, p_{t-\Delta})$ the public history at the beginning of $t > 0$ if no bidder has placed a valid bid up to t, and write $h_0 = \emptyset$ for the history at which the seller chooses the first reserve price.[1] Let H_t be the set of such histories. A (behavior) strategy for the seller specifies a Borel-measurable function $p_t : H_t \to P[0,1]$ for each $t = 0, \Delta, 2\Delta, \ldots$, where $P[0,1]$ is the space of Borel probability measures endowed with the weak* topology.[2] A (behavior) strategy for buyer i specifies a function $b_t^i : H_t \times [0,1] \times [0,1] \to P[0,1]$ for each $t = 0, \Delta, 2\Delta, \ldots$, where we assume that $b_t^i(h_t, p_t, v^i)$ is Borel-measurable in v^i, for all $h_t \in H_t$, and all $p_t \in [0,1]$, and that supp $b_t^i(h_t, p_t, v^i) \subset \{0\} \cup [p_t, 1]$, where "0" denotes no bid or an invalid bid.

We consider perfect Bayesian equilibria (PBE), and we will focus on symmetric weak Markov (or stationary) equilibria. Weak-Markov equilibria are defined as follows:

Definition 1. An equilibrium $(p, b) \in \mathcal{E}(\Delta)$ is a *weak-Markov (or stationary) equilibrium* if the buyers' strategies only depend on the reserve price announced for the current period.

3 Existence and Uniform Coase Conjecture

Following Ausubel and Deneckere (1989), we impose the following standard assumption. It is not needed for existence, but is used to extend Coasian conjecture to our auction setting.

Assumption 1. *There exist constants* $0 < M \leq 1 \leq L < \infty$ *and* $\alpha > 0$ *such that* $Mv^\alpha \leq F(v) \leq Lv^\alpha$ *for all* $v \in [0,1]$.

[1] We do not have to consider other histories because the game ends if someone places a valid bid.

[2] We slightly abuse notation by using p_t both for the seller's strategy and the announced reserve price at a given history.

In any equilibrium of the discrete time game, all buyers play pure strategies that are characterized by history-dependent cutoffs. This is captured by the following Lemma which establishes the "skimming property," an auction analog of a result by Fudenberg et al. (1985). Its proof is standard and thus omitted.

Lemma 1. *[Skimming Property] Let $(p, b) \in \mathcal{E}(\Delta)$. Then, for each $t = 0, \Delta, 2\Delta, \ldots$, there exists a function $\beta_t : H_t \times [0, 1] \to [0, 1]$ such that every bidder with valuation above $\beta_t(h_t, p_t)$ places a valid bid and every bidder with valuation below $\beta_t(h_t, p_t)$ waits if the seller announces reserve price p_t at history h_t.*

We now state the result:

Proposition 1.

1. *(Existence) A stationary equilibrium exists for every $r > 0$ and $\Delta > 0$.*
2. *(Uniform Coase Conjecture) Suppose Assumption 1 holds. For every $\varepsilon > 0$, there exists $\Delta_\varepsilon > 0$ such that for all $\Delta < \Delta_\varepsilon$, all $x \in [0, 1]$, and every symmetric stationary equilibrium (p, b) of the game with period length Δ and a truncated distribution $F(v|v \leq x)$ on $[0, x]$, the seller's profit associated with this equilibrium, $\Pi^\Delta(p, b|x)$, is bounded above by $(1 + \varepsilon)\Pi^E(x)$, where $\Pi^E(x)$ is the seller's profit from the efficient auction under this truncated distribution.*

The second part of the proposition shows that the seller's profit in every symmetric stationary equilibrium converges to the profit of the efficient auction.[3] The convergence is uniform, in the sense that $\Pi^\Delta(p, b|x)/\Pi^E(x) \to 1$ uniformly for all $x \in (0, 1]$.

[3]Notice that in contrast to the Coase conjecture for one buyer, Proposition 1.(ii) does not show that the initial reserve price p_0 converges to zero. This is in fact not the case in the auction setting as was noted by McAfee and Vincent (1997). However, reserve prices for $t > 0$ converge to zero which is sufficient for the convergence of equilibrium profits to the profit of an efficient auction—the counterpart of the Coase conjecture in the auction setting.

4 Proof

We adopt Ausubel and Deneckere (1989)'s notation and assume that the types of the bidders are i.i.d. draws from $U[0,1]$. We denote the type of buyer i by q^i. The valuation for each type is given by the function $v(q) := F^{-1}(q)$. Assumption 1 implies that the same condition also holds for $v(q)$ and corresponds to the assumption made in Definition 5.1 in Ausubel and Deneckere (1989). In the following we will use that F is continuous and strictly increasing (as in Ausubel and Deneckere (1989) we could relax this even further to general distribution functions but this is not necessary for the purpose of the present paper).[4] Since the proof of Proposition 1 follows closely the approach of Ausubel and Deneckere (1989), we only state proofs for the parts of the proof of Ausubel and Deneckere (1989) that need to be modified for the case of $n \geq 2$.

4.1 Proof of Proposition 1.(i)

In a weak-Markov equilibrium, the buyers' strategy can be described by a function $P : [0,1] \to [0,1]$. A bidder with type q^i places a valid bid if and only if the announced reserve price is smaller than $P(q^i)$. Given that v is strictly increasing, Lemma 1 implies that P is non-decreasing.

Also by Lemma 1, the posterior of the seller at any history is described by the supremum of the support, which we denote by q. If all buyers play according to P, the seller's (unconditional) continuation profit for given q is[5]

$$R(q) := \max_{y \in [0,q]} \int_y^q v(z) d\left[nz^{n-1} - (n-1)z^n\right] + P(y)\, n\,(q-y)y^{n-1} + e^{-r\Delta} R(y) \quad (1)$$

Let $Y(q)$ be the argmax correspondence and define $y(q) := \sup Y(q)$. Because

[4] In Ausubel and Deneckere (1989) the valuation is decreasing in the type. We define v to be increasing so that higher types have higher valuations.

[5] Dividing the RHS by q^n and replacing $R(y)$ by $y^n R(y)$ would yield the conditional continuation profit. The unconditional version is more convenient for the subsequent development.

the objective satisfies a single-crossing property, $Y(q)$ is increasing and hence single-valued almost everywhere. If $Y(q)$ is single-valued at q the seller announces a reserve price $S(q) = P(y(q))$ if the posterior has upper bound q.

The buyers' indifference condition for the case that $Y(q)$ is single-valued so that the seller does not randomize, is given by:

$$v(q) - P(q) = e^{-r\Delta}\left[v(q) - \frac{(y(q))^{n-1}}{q^{n-1}}S(q) - \frac{1}{q^{n-1}}\int_{y(q)}^{q} v(x)dx^{n-1}\right]. \tag{2}$$

If the seller randomizes over $Y(q)$ according to some probability measure μ, then

$$v(q) - P(q) = e^{-r\Delta}\left[v(q) - \int_{Y(q)}\left\{\frac{y^{n-1}}{q^{n-1}}P(y) + \frac{1}{q^{n-1}}\int_{y}^{q} v(x)dx^{n-1}\right\}d\mu(y)\right], \tag{3}$$

which may require that μ depends on $P(q)$.[6]

We will be looking for left-continuous functions R and P such that (1) and (2) are satisfied. If this is true for all $q \in [0, \bar{q}]$, then we say that (P, R) support a weak-Markov equilibrium on $[0, \bar{q}]$. The goal is to show the existence of a pair (P, R) that supports a weak-Markov equilibrium on $[0, 1]$. As in Ausubel and Deneckere (1989), we can show that the seller's continuation profit is Lipschitz-continuous in q.

Lemma 2. *[cf. Lemma A.2 in Ausubel and Deneckere (1989)] If (P, R) supports a weak-Markov equilibrium on $[0, \bar{q}]$, then R is increasing and Lipschitz continuous satisfying*

$$0 < R(q_1) - R(q_2) \leq n(q_1 - q_2)$$

for all $0 \leq q_2 < q_1 \leq \bar{q}$.

[6] In the following, we give details for the case that the seller does not randomize and refer to Ausubel and Deneckere (1989) for the discussion of randomization by the seller.

Proof. First, we show monotonicity:

$$\begin{aligned}
R(q_1) &= \int_{y(q_1)}^{q_1} v(z) d\left[nz^{n-1} - (n-1)z^n\right] + P(y(q_1))\, n\,(q_1 - y(q_1))(y(q_1))^{n-1} + e^{-r\Delta} R(y(q_1)) \\
&\geq \int_{y(q_2)}^{q_1} v(z) d\left[nz^{n-1} - (n-1)z^n\right] + P(y(q_2))\, n\,(q_1 - y(q_2))(y(q_2))^{n-1} + e^{-r\Delta} R(y(q_2)) \\
&> \int_{y(q_2)}^{q_2} v(z) d\left[nz^{n-1} - (n-1)z^n\right] + P(y(q_2))\, n\,(q_2 - y(q_2))(y(q_2))^{n-1} + e^{-r\Delta} R(y(q_2)) \\
&= R(q_2)
\end{aligned}$$

To show Lipschitz continuity, notice that the revenue from sales to types below q_2 in the continuation starting from q_1 is at most $R(q_2)$ and the revenue from types between q_2 and q_1 is bounded above by $P(q_1)(q_1^n - q_2^n)$.[7] Hence

$$\begin{aligned}
R(q_1) - R(q_2) &\leq P(q_1)(q_1^n - q_2^n) \\
&\leq (q_1^n - q_2^n) \\
&\leq n(q_1 - q_2)
\end{aligned}$$

□

Using this Lemma, we can show that an existence result for $[0, \bar{q}]$ can be extended to the whole interval $[0, 1]$.

Lemma 3. *[cf. Lemma A.3 in Ausubel and Deneckere (1989)] Suppose $(P_{\bar{q}}, R_{\bar{q}})$ supports a weak-Markov equilibrium on $[0, \bar{q}]$, then there exists (P, R) which supports a weak-Markov equilibrium on $[0, 1]$.*

[7] Suppose by contradiction that for the posterior $[0, q_1]$, the expected payment that the seller can extract from some type $q \in [q_2, q_1]$ is greater or equal than $P(q_1)$. In order to arrive at a history where the posterior is $[0, q_1]$, the seller must have used reserve price $P(q_1)$ in the previous period. But then all types in $[q, q_1]$ would prefer to bid in the previous period because they expect to make higher payments if they wait. This is a contradiction.

Proof. We extend $(R_{\bar{q}}, P_{\bar{q}})$ to some $[0, \bar{q}']$. Define

$$R_{\bar{q}'}(q) = \max_{0 \le y \le \min\{\bar{q}, q\}} \int_y^q v(z) d\left[nz^{n-1} - (n-1)z^n\right] + P_{\bar{q}}(y)\, n\,(q-y)y^{n-1} + e^{-r\Delta} R_{\bar{q}}(y)$$

with $y_{\bar{q}'}(q)$ as the supremum of the argmax correspondence. Moreover, we define $P_{\bar{q}'}(q)$ by

$$v(q) - P_{\bar{q}'}(q) = e^{-r\Delta}\left[v(q) - \frac{(y_{\bar{q}'}(q))^{n-1}}{q^{n-1}} P_{\bar{q}}(y_{\bar{q}'}(q)) - \frac{1}{q^{n-1}} \int_{y_{\bar{q}'}(q)}^q v(x) dx^{n-1}\right].$$

For $\bar{q}' = \min\left\{1, \sqrt[n]{\bar{q}^n + (1 - e^{-r\Delta}) R_{\bar{q}}(\bar{q})}\right\}$, the constraint in the maximization in the definition of $R_{\bar{q}'}(q)$ is not binding and moreover

$$R_{\bar{q}'}(q) = \max_{0 \le y \le q} \int_y^q v(z) d\left[nz^{n-1} - (n-1)z^n\right] + P_{\bar{q}'}(q)\, n\,(q-y)y^{n-1} + e^{-r\Delta} R_{\bar{q}'}(y)$$

For $y \in [\bar{q}, q]$ we have

$$\begin{aligned}
&\int_y^q v(z) d\left[nz^{n-1} - (n-1)z^n\right] + P_{\bar{q}'}(q)\, n\,(q-y)y^{n-1} + e^{-r\Delta} R_{\bar{q}'}(y) \\
\le& q^n - y^n + e^{-r\Delta} R_{\bar{q}'}(q) \\
\le& (1 - e^{-r\Delta}) R_{\bar{q}}(\bar{q}) + e^{-r\Delta} R_{\bar{q}'}(q) \\
\le& (1 - e^{-r\Delta}) R_{\bar{q}'}(q) + e^{-r\Delta} R_{\bar{q}'}(q) \\
\le& R_{\bar{q}'}(q).
\end{aligned}$$

In the first step, we have used that the payments $v(z)$ and $P_{\bar{q}'}(q)$ are less than or equal to one. In the second step, we have used that $\bar{q}' = \min\left\{1, \sqrt[n]{\bar{q}^n + (1 - e^{-r\Delta}) R_{\bar{q}}(\bar{q})}\right\}$; since $\bar{q} \le y \le q \le \bar{q}'$, this implies $q^n - y^n \le (1 - e^{-r\Delta}) R_{\bar{q}}(\bar{q})$. The third step uses $R_{\bar{q}}(\bar{q}) = R_{\bar{q}'}(\bar{q})$ and that $R_{\bar{q}'}$ is increasing. Thus $(P_{\bar{q}'}, R_{\bar{q}'})$ supports a weak-Markov equilibrium on $[0, \bar{q}']$. Since $R_{\bar{q}}(\bar{q}) > 0$, a finite number of repetitions suffices to extend $(P_{\bar{q}}, R_{\bar{q}})$ to the entire interval $[0, 1]$. □

To complete the proof, we follow Ausubel and Deneckere (1989) by replacing the lower tail distribution on the interval $[0, \bar{q}]$ by a uniform distribution. For the uniform distribution, a weak-Markov equilibrium can be constructed explicitly. In the auction case, this has been shown by McAfee and Vincent (1997). Therefore, Lemma 3 implies that for the modified distribution with a uniform part at the lower end, a weak-Markov equilibrium exists. The final step is to show that the functions (P, R) that support the equilibrium for the modified distribution converge to functions that support a weak-Markov equilibrium for the original distribution as $\bar{q} \to 0$.

Proof. [Proof of Proposition 1.(i)] As in Ausubel and Deneckere (1989), we consider a sequence of valuation functions

$$v_\eta(q) = \begin{cases} v(q), & \text{if } q \geq \frac{1}{\eta} \\ v\left(\frac{1}{\eta}\right) \eta q, & \text{otherwise.} \end{cases}$$

This corresponds to the original distribution except that on the interval $[0, 1/\eta]$, we have made the distribution uniform. McAfee and Vincent (1997) show that there exist $(\tilde{P}_{1/\eta}, \tilde{R}_{1/\eta})$ that support a weak-Markov equilibrium on $[0, 1/\eta]$. Hence, by Lemma 3, for each $\eta = 1, 2, \ldots$, there exists a pair (P_η, R_η) that supports a weak-Markov equilibrium on $[0, 1]$. As in Ausubel and Deneckere (1989), we can assume that P_η converges point-wise for all rationals to some function $\Phi(s)$, $s \in \mathbb{Q} \cap [0, 1]$ and taking left limits we can extend this limit to a non-decreasing, left-continuous function $P : [0, 1] \to [0, 1]$. Also, by Lemma 2, after taking a sub-sequence, we may assume that (R_n) converges uniformly to a continuous function R. We have to show that (P, R) supports a weak-Markov equilibrium for v. But given Lemma 2 and 3, only minor modifications are needed to apply the proof of Theorem 4.2 from Ausubel and Deneckere (1989). □

4.2 Proof of Proposition 1.(ii)

Before we begin with the proof, we note that in contrast to the case of one buyer analyzed by Ausubel and Deneckere (1989), the first reserve price in a continuation game where the seller's posterior is v_t need not converge to zero as $\Delta \to 0$.[8] Nevertheless, we obtain the Coase conjecture because prices fall arbitrarily quickly as $\Delta \to 0$. On the buyer side, the strategy is described by a cutoff for the reserve price. A buyer places a bid if and only if the current reserve price is below the cutoff. The Markov property of the buyer's strategy implies that the cutoff only depends on the buyer's type, it is independent of time and of the history of previous reserve prices. As $\Delta \to 0$, the equilibrium cutoff of a buyer with type v converges to the payment that this type would make in a second-price auction without reserve price. Also reserve prices decline arbitrarily quickly so that the delay of the allocation vanishes for all buyers as $\Delta \to 0$. Therefore, the seller's profit converges to the profit of an efficient auction.

We want to show that the profit of the seller in any weak-Markov equilibrium of a subgame that starts with the posterior $[0, q]$, converges (uniformly over q) to $\Pi^E(q)$ as $\Delta \to 0$. The proof consists of two main steps. The first step shows that for any type $\xi \in [0, 1]$, any $\Delta > 0$, and any weak-Markov equilibrium supported by some pair (P, R), the expected payment that the seller can extract from type ξ is bounded by $\xi^{n-1}P(\xi)$. We prove this by showing that the expected payment conditional on winning is bounded by $P(\xi)$.

Lemma 4. *Let (P, R) support a weak-Markov equilibrium in the game for $\Delta > 0$. Suppose that in this equilibrium, type $\xi \in [0, 1]$ trades in period t, let the posterior in period t be $q_t \geq \xi$, and denote the marginal type in period t by $q_t^+ \leq \xi$. Then we have*

$$P(\xi) \geq \int_{q_t^+}^{\xi} v(x) \frac{dx^{n-1}}{\xi^{n-1}} + \frac{\left(q_t^+\right)^{n-1}}{\xi^{n-1}} P(q_t^+), \qquad \forall \xi \in [0, 1],$$

[8] For the uniform distribution, this was already noted by McAfee and Vincent (1997).

and hence

$$R(q) \leq \int_0^q P(x)\, dx^n, \qquad \forall q \in (0,1].$$

Proof. For $q_t^+ = \xi$ the RHS of the first inequality becomes $P(q_t^+) = P(\xi)$. Hence it suffices to show that

$$\int_q^\xi v(x) dx^{n-1} + q^{n-1} P(q)$$

is increasing in q. For $q > \hat{q}$ we have

$$\begin{aligned}
&\int_q^\xi v(x) dx^{n-1} + q^{n-1} P(q) - \int_{\hat{q}}^\xi v(x) dx^{n-1} - \hat{q}^{n-1} P(\hat{q}) \\
=& q^{n-1} P(q) - \hat{q}^{n-1} P(\hat{q}) - \int_{\hat{q}}^q v(x) dx^{n-1}
\end{aligned}$$

Using (2), we have

$$\begin{aligned}
& q^{n-1} P(q) - \hat{q}^{n-1} P(\hat{q}) \\
=& \left(1 - e^{-r\Delta}\right) q^{n-1} v(q) + e^{-r\Delta} \int_{y(q)}^q v(x) dx^{n-1} + e^{-r\Delta} \left(y(q)\right)^{n-1} P(y(q)) \\
& - \left(1 - e^{-r\Delta}\right) \hat{q}^{n-1} v(\hat{q}) - e^{-r\Delta} \int_{y(\hat{q})}^{\hat{q}} v(x) dx^{n-1} - e^{-r\Delta} \left(y(\hat{q})\right)^{n-1} P(y(\hat{q})) \\
=& \left(1 - e^{-r\Delta}\right) \left(q^{n-1} v(q) - \hat{q}^{n-1} v(\hat{q})\right) + e^{-r\Delta} \left(\left(y(q)\right)^{n-1} P(y(q)) - \left(y(\hat{q})\right)^{n-1} P(y(\hat{q}))\right) \\
& + e^{-r\Delta} \int_{\hat{q}}^q v(x) dx^{n-1} - e^{-r\Delta} \int_{y(\hat{q})}^{y(q)} v(x) dx^{n-1}
\end{aligned}$$

and hence

$$
\begin{aligned}
& q^{n-1}P(q) - \hat{q}^{n-1}P(\hat{q}) - \int_{\hat{q}}^{q} v(x)dx^{n-1} \\
=& \left(1 - e^{-r\Delta}\right)\left(q^{n-1}v(q) - \hat{q}^{n-1}v(\hat{q})\right) + e^{-r\Delta}\left((y(q))^{n-1}P(y(q)) - (y(\hat{q}))^{n-1}P(y(\hat{q}))\right) \\
& - \left(1 - e^{-r\Delta}\right)\int_{\hat{q}}^{q} v(x)dx^{n-1} - e^{-r\Delta}\int_{y(\hat{q})}^{y(q)} v(x)dx^{n-1} \\
=& e^{-r\Delta}\left((y(q))^{n-1}P(y(q)) - (y(\hat{q}))^{n-1}P(y(\hat{q})) - \int_{y(\hat{q})}^{y(q)} v(x)dx^{n-1}\right) \\
& + \left(1 - e^{-r\Delta}\right)\int_{\hat{q}}^{q} v'(x)x^{n-1}dx
\end{aligned}
$$

Proceeding inductively, we get

$$
q^{n-1}P(q) - \hat{q}^{n-1}P(\hat{q}) - \int_{\hat{q}}^{q} v(x)dx^{n-1} = \sum_{k=0}^{\infty} e^{-k\Delta}\left(1 - e^{-r\Delta}\right)\int_{y^k(\hat{q})}^{y^k(q)} v'(x)x^{n-1}dx > 0,
$$

where $y^k(\cdot)$ denotes the function obtained by applying $y(\cdot)$ k times. This shows the first inequality.

For the second inequality, notice that the RHS of the first inequality is the payment that the seller can extract from type ξ if ξ wins the auction. This is bounded by $P(\xi)$ as the first inequality shows. The seller's profit if the posterior at time t is q, therefore satisfies

$$
R(q) \leq \int_{0}^{q} e^{-r(T(x)-t)}P(x)dx^{n},
$$

where $T(x)$ denotes the trading time of type x in the weak-Markov equilibrium. This implies the second inequality. □

For the second step, fix the distribution and the corresponding function v and define $v_x : [0,1] \to [0,1]$ for all $x \in (0,1]$.

$$
v_x(q) := \frac{v(qx)}{v(x)}.
$$

Using Helly's selection theorem, we can extend this definition to $x = 0$, by taking the a.e.-limit of a subsequence of functions v_x. Denote by $\mathcal{E}^{wM}(\Delta, x)$ the weak-Markov equilibria of the game with discount factor Δ and distribution given by v_x where $x \to 0$. Slightly abusing notation we write $(P, R) \in \mathcal{E}^{wM}(\Delta, x)$ for a weak-Markov equilibrium that is supported by functions (P, R). We show that there is an upper bound for $P(1)$ that converges to the expected payment in a second price auction without reserve price as $\Delta \to 0$, and the convergence is uniform over x.

Lemma 5. *Fix $v(\cdot)$. For all $\varepsilon > 0$, there exists $\Delta_\varepsilon > 0$ such that for all $\Delta \leq \Delta_\varepsilon$, all $x \in [0, 1]$, and all $(P, R) \in \mathcal{E}^{wM}(\Delta, x)$,*

$$P(1) \leq \int_0^1 v_x(s)\, ds^{n-1} + \varepsilon.$$

Proof. Suppose not. Then there exist sequences $\Delta_m \to 0$ and $x_m \to \bar{x}$ such that for all $m \in \mathbb{N}$, there exist equilibria $(P_m, R_m) \in \mathcal{E}^{wM}(\Delta_m, x_m)$ such that for all m,

$$P_m(1) > \int_0^1 v_{x_m}(s)\, ds^{n-1} + \varepsilon.$$

By a similar argument as in the proof of Theorem 4.2 of Ausubel and Deneckere (1989), we can construct a limiting pair $(\overline{P}, \overline{R})$, where $\overline{P}$ is left-continuous and non-decreasing, P_m converges point-wise to $\overline{P}$ for all rationals, and R_m converges uniformly to $\overline{R}$. Obviously, we have

$$\overline{P}(1) \geq \int_0^1 v_{\bar{x}}(s)\, ds^{n-1} + \varepsilon.$$

Left-continuity implies that there exists $\bar{q} < 1$ such that

$$\overline{P}(\bar{q}) \geq \int_0^{\bar{q}} v_{\bar{x}}(s)\, ds^{n-1} + \frac{\varepsilon}{2}. \tag{4}$$

Using an argument from the proof of Theorem 5.4 in Ausubel and Deneckere

(1989), we can show that

$$\overline{R}(1) \geq \int_{\bar{q}}^{1} \overline{P}(s)\, ds^n + \Pi^E(\bar{q}) \geq \Pi^E(1) + (1-\bar{q})\frac{\varepsilon}{2},$$

where we have used (4) to show the second inequality. Hence, we have

$$R_m(1) \to \overline{R}(1) \geq \Pi^E(1) + (1-\bar{q})\frac{\varepsilon}{2}. \tag{5}$$

But this implies that there must exist a type $\hat{q} > 0$, a time $t > 0$, and $\bar{m}$ such that for all $m > \bar{m}$,

$$T_m(\hat{q}) \geq t.$$

where $T_m(\cdot)$ is the trading time function in the weak-Markov equilibrium supported by (P_m, R_m). To see this, note that delay for low types is needed to increase the seller's revenue beyond the revenue from an efficient auction.

With this observation, we can conclude the proof using a similar argument as in Case I of the proof of Theorem 5.4 in Ausubel and Deneckere (1989). From Lemma 4 we know that the maximal expected payment conditional on winning that a buyer of type q has to make in equilibrium is given by $P_m(q)$. This implies that

$$R_m(1) \leq \int_{\hat{q}}^{1} P_m(z) dz^n + e^{-rt} R_m(\hat{q}).$$

In the limit we have

$$\overline{R}(1) \leq \int_{\hat{q}}^{1} \overline{P}(z) dz^n + e^{-rt}\overline{R}(\hat{q}). \tag{6}$$

On the other hand, the same argument that we used to obtain (5) yields

$$\overline{R}(1) \geq \int_{0}^{1} \overline{P}(z) dz^n. \tag{7}$$

Combining (6) and (7) we get

$$\int_0^{\hat{q}} \overline{P}(z)dz^n \leq e^{-rt}\overline{R}(\hat{q}),$$

which implies

$$\overline{R}(\hat{q}) > \int_0^{\hat{q}} \overline{P}(z)dz^n,$$

since $t > 0$. But Lemma 4 implies the opposite inequality which is a contradiction. □

Using this Lemma, we can show that for a given $v(\cdot)$, the difference between the continuation profit at $[0, q]$ and $\Pi^E(q)$, divided by $v(q)$ converges uniformly to zero.

Lemma 6. *Fix $v(\cdot)$. For all $\varepsilon > 0$, there exists $\Delta_\varepsilon > 0$ such that for all $\Delta \leq \Delta_\varepsilon$, all $x \in [0, 1]$, and all $(P, R) \in \mathcal{E}^{wM}(\Delta, 1)$,*

$$\frac{R(x)}{x^n} - \Pi^E(v(x)) \leq \varepsilon v(x).$$

Proof. The statement of the Lemma is equivalent to the statement that for all $\varepsilon > 0$, there exists $\Delta_\varepsilon > 0$ such that for all $\Delta \leq \Delta_\varepsilon$, all $x \in [0, 1]$, and all $(P, R) \in \mathcal{E}^{wM}(\Delta, x)$,

$$R(1|v_x) - \Pi^E(1|v_x) \leq \varepsilon. \tag{8}$$

This equivalence holds because truncating and rescaling the function $v(\cdot)$ leads to the following transformations:

$$\frac{R(x|v)}{x^n} = v(x)R(1|v_x),$$
$$\Pi^E(v(x)) = v(x)\Pi^E(1|v_x).$$

To show (8), we combine Lemmas 4 and 5, and use that $P(z|v_x) = v_x(z)P(1|v_{z\cdot x})$ to

get for all $x \in [0,1]$,

$$\begin{aligned}
R(1|v_x) &\leq \int_0^1 P(z|v_x)dz^n \\
&= \int_0^1 v_x(z)P(1|v_{x\cdot z})dz^n \\
&\leq \int_0^1 v_x(z)\left(\int_0^1 v_{x\cdot z}(s)ds^{n-1} + \varepsilon\right)dz^n \\
&= \int_0^1 \left(\int_0^1 v_x(sz)ds^{n-1}\right)dz^n + \varepsilon\int_0^1 v_x(z)dz^n \\
&\leq \int_0^1 \left(\int_0^z v_x(s)\frac{ds^{n-1}}{z^{n-1}}\right)dz^n + \varepsilon \\
&= \Pi^E(1|v_x) + \varepsilon
\end{aligned}$$

□

This allows us to complete the proof of Proposition 1.(ii).

Proof of Proposition 1.(ii) Translated into the notation of the paper, Lemma 6 implies that for a given distribution function F, for all $\tilde{\varepsilon} > 0$, there exists $\Delta_{\tilde{\varepsilon}} > 0$ such that for all $\Delta \leq \Delta_{\tilde{\varepsilon}}$, all $v \in [0,1]$, and all weak-Markov equilibria $(p,b) \in \mathcal{E}^{wM}(\Delta)$, we have

$$\Pi^{\Delta}(p,b|v) \leq \Pi^E(v) + \tilde{\varepsilon}v.$$

Recall that by Assumption 1, there exist $0 < M \leq 1 \leq L < \infty$ and $\alpha > 0$ such that $Mv^\alpha \leq F(v) \leq Lv^\alpha$ for all $v \in [0,1]$. This implies that the rescaled truncated distribution

$$\tilde{F}_x(v) := \frac{F(vx)}{F(x)},$$

for all $v \in [0,1]$ is dominated by a function that is independent of x:

$$\tilde{F}_x(v) \leq \frac{Lv^\alpha x^\alpha}{Mx^\alpha} = \frac{L}{M}v^\alpha.$$

Next, we observe that the revenue of the efficient auction can be written in terms of the rescaled expected value of the second-highest order statistic of the rescaled distribution:

$$\Pi^E(v) = \int_0^1 vs\tilde{F}_v^{(n-1:n)}(s)ds.$$

If we define $\hat{F}(v) := \min\left\{1, \frac{L}{M}v^\alpha\right\}$ and $B := \int_0^1 s\hat{F}^{(n-1:n)}(s)ds$, then given $\tilde{F}_x(v) \leq \frac{L}{M}v^\alpha$ we can apply Theorem 4.4.1 in David and Nagaraja (2003) to obtain $\Pi^E(v) \geq Bv > 0$ for all $v \in [0,1]$. If we chose $\tilde{\varepsilon}$ sufficiently small we have

$$\begin{aligned} & \tilde{\varepsilon} \leq B\varepsilon, \\ \iff \quad & \tilde{\varepsilon}v \leq B\varepsilon v, \\ \implies \quad & \tilde{\varepsilon}v \leq \varepsilon\Pi^E(v), \\ \iff \quad & \Pi^E(v) + \tilde{\varepsilon}v \leq (1+\varepsilon)\Pi^E(v). \end{aligned}$$

This implies that

$$\Pi^\Delta(p, b|v) \leq (1+\varepsilon)\Pi^E(v)$$

for all $\Delta \leq \Delta_\varepsilon := \Delta_{\tilde{\varepsilon}}$ for $\tilde{\varepsilon}$ sufficiently small.□

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